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This paper examines the magneto hydrodynamic two-phase blood (Casson fluid) flow in a vessel with heat conduction between blood and particles. The temperature of both phases is also considered. The model for the flow under consideration is formulated in terms of partial differential equations. Then the classical model is generalized by utilizing the Caputo fractional order derivative. The generalized equations are then non-dimensionalized by using appropriate dimensionless variables. The exact dimensionless solutions are obtained via the joint application of Laplace & Hankel integral transforms. The influence of various embedded parameters on both the velocities (blood and magnetic particles) and the temperature distribution are presented graphically. It is worth noting that the particle and blood velocities decrease for increasing the values of magnetic parameter ($H$) which is useful to control the blood flow during magnetic therapy (for treating pain, such as the back, foot, or joint pain) and surgeries. It is worth noting that fractional model better describes the flow behavior than classical model by providing various integral curves as shown in Fig.

Keywords : Casson fluid, two-phase blood flow, heat conduction, magneto hydrodynamic, Caputo fractional order derivative, Laplace & Hankel integral transforms

1. Introduction

Bio magnetic fluid dynamics (BFD) is the new research area in fluid dynamics that studies the bio fluids flow in the presence of a magnetic field. In medical sciences and bioengineering, bio magnetic fluid dynamics have several applications, specifically in targeting drug delivery [1]. It reduces blood loss in the process of surgeries and cures cancer tumors via hyperthermia [2]. It can also be applied to improve magnetic devices for the separation of cells [3]. In BFD the most suitable fluid is blood. Blood is the suspension of RBC’s or erythrocytes and plasma (watery fluid) [4]. Because of the presence of erythrocytes, blood behaves like a bio magnetic fluid. Erythrocytes are the red blood cells that contain hemoglobin which plays the role of magnetic particles and carrying oxygen to the body. Due to the negative charge, erythrocytes create a magnetic field on the walls of the vessel. Usually, targeted drug delivery is used to magnetize the blood by adding nanoparticles. The first bio magnetic fluid model was suggested by Haik et al. [5]. They studied the models for ferrohydrodynamics (FHD) and BFD comparatively and assumed that the flow is under the magnetic field's impact, but the induced magnetic field is not considered. Blood is an example of Casson fluid defined as the shear thinning liquids and is accepted to have an infinite viscosity at zero deformation [6]. When the yield stress is applied to fluids, and the shear stresses less than the yield stresses, the fluids behave like a solid. However, when the shear stresses are larger than the yield stress applied, it begins to flow. The flow of blood with larger shear rates through arteries of large diameter shows Newtonian behavior, but blood flow with low rate of shear through small arteries shows non-Newtonian behavior [7, 8]. The flow of blood with a low shear rate through narrow arteries shows a Casson fluid [9-11]. The mathematical model for the flow of blood through narrow arteries with low shear rates has been presented by many researchers [12-14]. For narrow capillaries of diameter 130 µm-1000 µm, the Casson fluid model is appropriate and discussed

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The heat transfer can be noticed in various fields of industry. It has many applications in the gas and oil industry, chemical industry, and pharmaceutical industry. Radiation, conduction, and convection are the three different modes in which heat transfers [17]. The transfer of heat in the form of gas or liquid to colder matter from warmed matter is called convection [18]. The uses of heat transmission are established in electrical devices, thermal power stations, engineering, automotive engineering, insulation, and climate control [19]. The unsteady Casson fluid flow and heat transfer over a moving flat plate is considered by Mustafa et al. [20]. Hayat et al. [21] found the solutions using the Homotopy analysis method to discuss the Casson fluid in stretching cylinder with nanoparticles and thermal radiations. In the above study, the researchers studied the classical Casson fluid models. The classical derivatives were used, and fractional-order derivatives were ignored because of their complexities [19-21].

Fractional calculus is the generalization of ordinary calculus in which the non-integer order derivative and integrals have been discussed. Fractional calculus was developed three hundred years ago by Laplace, Fourier, Abel, and Liouville [22]. Fractional calculus is applied in science, biosciences, mathematics, electrochemistry, physics, engineering, and technology [23]. Moreover, fractional derivatives have several applications in fluid mechanics, mathematical biology, viscoelasticity, electrochemistry, and signal processing [24-28]. Therefore, the significance of fractional order derivatives cannot be ignored. Bakhti et al. [29] considered the pulsatile flow of blood through stenosed arteries because of pressure gradient using fractional-order derivatives. Ali et al. [30] examined blood flow as a Casson fluid through a horizontal tube and considering the fractional-order derivative. Shah al. [31] considered the fractional-order derivative to investigate the flow of blood with magnetic particles and obtained the solutions by the joint use of Laplace & Hankel transforms. The influence of different parameters on the blood flow, particles velocity, temperature of blood and particles is graphically studied.

2. Mathematical Formulation

In this paper, the non-Newtonian flow of blood with a magnetic particle is studied in a vessel of radius $R_0$ as shown in Fig. 1. We assume that the magnetic particles are equally dispersed in the blood. The vessel is taken along the $z$-axis, and the radial axis is $r$. The magnetic field acts perpendicularly on the flow direction. Because of the smallness of the magnetic Reynolds number (Re), the induced magnetic field is neglected [34]. At the time $t = 0$, the vessel, particles, and blood are assumed to be at rest.

Under the impact of a transverse applied magnetic field,
the equation of blood velocity in a vessel is given by [35, 36]:

\[
\frac{\partial \omega (r, t)}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial^2 \omega (r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial \omega (r, t)}{\partial r} \right) 
+ \frac{KN}{\rho} \{ \omega_i (r, t) - \omega (r, t) \} - \frac{\sigma B^2}{\rho} \omega (r, t).
\]

(1)

The oscillatory pressure gradient \( \frac{\partial p}{\partial z} \) is given as [37]:

\[
- \frac{\partial p}{\partial z} = b_0 + b_1 \cos(\omega t), \quad t > 0.
\]

(2)

Where \( \omega \) is the velocity of blood, \( \omega_i \) is the particle velocity, \( \rho \) is the density of blood, \( \mu \) is the dynamic viscosity, \( \beta = \frac{\mu_0 \sqrt{2} \pi r}{r} \) is the parameter for Casson fluid in which \( \mu_0, \tau, \text{and } \pi \) are the plastic dynamic viscosity, yield stress and the critical value of this product based on a non-Newtonian fluid model respectively. The third term on the right-hand side of Eq. (1) i.e \( \frac{KN}{\rho} \{ \omega_i (r, t) - \omega (r, t) \} \) shows the forces between blood and magnetic particles because of the relative motion. Where \( K \) and \( N \) are the Stokes constant, and the number of magnetic particles per unit volume, \( \sigma \) is the electrical conductivity, \( B_0 \) is the applied magnetic field, \( b_0 \) is the systolic and \( b_1 \) is the diastolic amplitudes of pressure gradient and \( \omega \) is the frequency [38, 39].

From Newton’s 2nd law, the governing equation of particles (magnetic) is given in the form [40]:

\[
m \frac{\partial \omega_i}{\partial t} = K \{ \omega_i (r, t) - \omega (r, t) \}.
\]

(3)

Where \( m \) and \( \omega_i \) are the mass and velocity of particles, respectively.

The temperature equation of the blood is given by [41, 42]:

\[
\frac{\partial T (r, t)}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T (r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T (r, t)}{\partial r} \right) 
- \frac{4 \zeta^2}{\rho C_p} (T - T_s) + \frac{\mu}{\rho C_p} \left( \frac{\partial \omega}{\partial r} \right)^2 + \frac{\rho C_p}{\rho C_p} (T - T_s).
\]

(4)

where \( T \) represents the blood temperature, \( k \) and \( C_p \) represent the thermal conductivity and specific heat at constant pressure respectively, \( \zeta^2 \) is the Planck mean linear absorption coefficient, \( \left( \frac{\partial \omega}{\partial r} \right)^2 \) represents the viscous dissipation, \( \rho \) and \( C_p \) represent the density and specific heat of particles respectively, \( \gamma_r \) is the temperature relaxation time parameter and \( T_s \) is the temperature of particles.

The temperature equation of particles is given by [43]:

\[
\frac{\partial T (r, t)}{\partial t} = \frac{1}{\gamma_r} (T - T_s), \quad t > 0, \ r \in (0, R_b).
\]

(5)

The corresponding initial conditions and boundary conditions are given by:

\[
\begin{align*}
&w(r, 0) = 0, \quad w_i(r, 0) = 0, \\
&w(R_b, t) = 0, \quad w_i(R_b, t) = 0, \\
&T(r, 0) = T_s, \quad T(R_b, t) = T_u, \\
&T_s = T_s, \quad T_u = T_u.
\end{align*}
\]

(6)

Multiplying each term of Eqs. (1), (3), (4), and (5) by \( \lambda = \frac{R_b \rho}{b_0} \) to convert from their classical derivative to the Caputo fractional derivative

\[
\lambda^\alpha D^\alpha_{\gamma_r} w(r, t) = \frac{\lambda}{\rho} \left( b_0 + b_1 \cos(\omega t) \right) + \lambda \nu \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial^2 w(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial w(r, t)}{\partial r} \right)
+ \lambda \frac{KN}{\rho} \{ \omega_i (r, t) - \omega (r, t) \} - \frac{\lambda \sigma B^2}{\rho} \omega (r, t),
\]

(7)

\[
\lambda^\alpha D^\alpha_{\gamma_r} w_i(r, t) = \frac{\lambda K}{m} \{ w_i(r, t) - \omega_i (r, t) \},
\]

(8)

\[
\lambda^\alpha D^\alpha_{\gamma_r} T(r, t) = \frac{k \lambda}{\rho C_p} \left( \frac{\partial T (r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T (r, t)}{\partial r} \right) - \frac{4 \lambda \zeta^2}{\rho C_p} (T - T_s) + \frac{\lambda \nu}{C_p} \left( \frac{\partial \omega}{\partial r} \right)^2 + \frac{\lambda \rho C_p}{\rho C_p} (T - T_s),
\]

(9)

\[
\lambda^\alpha D^\alpha_{\gamma_r} T_i(r, t) = \frac{\lambda}{\gamma_r} (T - T_s),
\]

(10)

where;

\[
D^\alpha_{\gamma_r} f(r, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial f(r, \tau)}{\partial \tau} d\tau,
\]

(11)

is the Caputo time derivative of fractional order \( \alpha \) [44]:

By introducing the appropriate non dimensional variables:

\[
r' = \frac{r}{R_b}, \quad t' = \frac{t}{\lambda}, \quad \omega' = \frac{\omega}{\omega_b}, \quad \nu' = \frac{\nu}{\nu_b}, \quad \beta' = \frac{\beta}{\beta_b}, \quad \alpha' = \lambda \omega, \\
T' = \frac{T - T_s}{T_u - T_s}, \quad \gamma_r' = \frac{T_s - T_u}{T_u - T_s}.
\]

(12)

After introducing the dimensionless variables and
dropping the (*) notation Eqs. (7-10), take the form

\[ D_0^+ w(r, t) = [b_0 + b_1 \cos(\alpha t)] + \beta \frac{\partial^2 w(r, t)}{\partial r^2} + \frac{\beta \partial w(r, t)}{r} + R \{ w(r, t) - w'(r, t) \} - H w(r, t). \]  

(13)

\[ GD_0^+ w_i(r, t) = \{ w(r, t) - w_i(r, t) \}. \]  

(14)

\[ P D_s^+ T(r, t) = \left( \frac{\partial^2 T(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r, t)}{\partial r} \right) - 4 R_s T(r, t) + B_i \left( \frac{\partial w}{\partial r} \right)^2 + R (T(r, t) - T_i(t)). \]  

(15)

\[ D_i^+ T_i(r, t) = \gamma (T - T_i). \]  

(16)

\[ \beta_i = \frac{1}{\text{Re}} (1 + \frac{1}{\beta_i}) \quad \text{Re} = \frac{R_0^2}{\lambda V}, \quad R = \frac{\lambda KN}{\rho}, \quad H = \sqrt{\frac{1}{\alpha \sigma B_0^2}} \rho, \quad G = \frac{m}{K \lambda}, \quad P_c = \frac{\rho C_p R_o^2}{k \lambda}, \quad R_o = \frac{\alpha^2 R_o^2}{k}, \quad B_i = \frac{\mu B_0^2}{k(T_0 - T_c)}, \]

\[ R_i = \frac{\rho P_c R_o^2}{k \gamma_T} \quad \text{and} \quad \gamma = \frac{\lambda}{\gamma_T}. \]

Where Re is the Reynold number, \( R \) is the concentration parameter of particle, \( H^2 \) is the magnetic parameter, \( G \) is the parameter for particle mass. \( P_c \) and \( R_o \) show the Peclet number and radiation absorption parameter respectively. \( B_0 \) and \( R_o \) represent the Brinkman number and particle concentration parameter, and \( \gamma \) is the temperature relaxation time-parameter.

The initial & boundary conditions become:

\[
\begin{align*}
   w(r, 0) &= 0, \quad w_i(r, 0) = 0, \\
   w(1, t) &= 0, \quad w_i(1, t) = 0, \\
   T(r, 0) &= 0, \quad T_i(1, t) = 1, \\
   T_i(r, 0) &= 0, \quad T_i(1, t) = 1, \\
   \left. \frac{\partial w}{\partial r} \right|_{r=0} &= 0
\end{align*}
\]

(17)

### 3. Solution of the Problem

To find the solution to the problem, we use the Laplace and Hankel transforms jointly.

#### 3.1. Solution of Velocity Equation

By applying the Laplace transform to Eq. (13) and Eq. (14), respectively, and using the initial conditions, we get:

\[ s^m w(r, s) = \frac{b_0}{s} + \frac{b_1}{s^2 + \omega} + \beta \left( \frac{s^m \bar{w}(r, s)}{s^2 + \omega} + \frac{1}{r} \frac{\partial \bar{w}(r, s)}{\partial r} \right) \]

(18)

\[ + R \bar{w}(r, s) - \bar{w}(r)(R + H), \]

(19)

With the boundary conditions in the transformed form:

\[ \bar{w}(1, s) = 0, \quad \bar{w}_i(1, s) = 0. \]

(20)

Using Eq. (19) into Eq. (18), we get:

\[ \bar{w}(r, s) = \frac{b_0}{s} + \frac{b_1}{s^2 + \omega} + \beta \left( \frac{s^m \bar{w}(r, s)}{s^2 + \omega} + \frac{1}{r} \frac{\partial \bar{w}(r, s)}{\partial r} \right) \]

(21)

by using the finite Hankel transform of zero-order [45, 46] in Eq. (21) and applying the boundary conditions from Eq. (20), we get:

\[ \bar{w}(r, s) = \frac{b_0}{s} + \frac{b_1}{s^2 + \omega} + \beta \left( \frac{s^m \bar{w}(r, s)}{s^2 + \omega} + \frac{1}{r} \frac{\partial \bar{w}(r, s)}{\partial r} \right) \]

(22)

Where \( \bar{w}(r, s) = \int_0^1 \bar{w}(r, s) J_n(r_n) dr_n \) is the finite Hankel transform of zero-order of \( \bar{w}(r, s) \), and \( r_n (n = 1, 2, 3, 4, \ldots) \) are the positive roots of \( J_n(x) = 0 \), \( J_n(\cdot) \) is the Bessel function of the first kind of zero order. Eq. (22) can be written in the more suitable form as:

\[ \bar{w}(r, s) = \frac{b_0}{s} + \frac{b_1}{s^2 + \omega} + \beta \left( \frac{s^m \bar{w}(r, s)}{s^2 + \omega} + \frac{1}{r} \frac{\partial \bar{w}(r, s)}{\partial r} \right) \]

(23)

Where:

\[ a_m = \frac{b_n - c_n}{2G}, \quad a_2 = \frac{b_n + c_n}{2G}, \quad b_n = 1 + G(R + H) + \beta r_n^2 G, \]

\[ c_n = \sqrt{b_n^2 - 4G(H + \beta r_n^2)}. \]

Using the particular functions to get the inverse Laplace transform of Eq. (23):

\[ L^{-1} \left[ \frac{1}{s^m + m} \right] = F_\alpha(-m, t) = \sum_{n=0}^\infty \frac{(-m)^n t^{n+1}}{\Gamma(a(n+1))}, \]

(24)

\[ L^{-1} \left[ \frac{s^m}{s^m + m} \right] = R_\alpha(-m, t) = \sum_{n=0}^\infty \frac{(-m)^n t^{n+1}}{\Gamma(a(n+1) - \eta)}, \]

(25)
Where \( F_n(x) \) and \( R_n(x) \) are the Robotnov-Hartley’s and the Lorenzo-Hartley’s functions respectively \([47]\), we get:

\[
w(r,t) = \sum_{n=1}^{\infty} \frac{J_n(r_R)}{r_R J_n'(r_R)} \left[ F_n(t) + F_{2n}(t) \right],
\]

(26)

\[
F_n(t) = \frac{b_n}{c_n} \left( 1 - \frac{G a_n}{s+\omega} \right) R_{n-1}(-a_n t) - \frac{G a_n}{s+\omega} R_{n-2}(-a_n t),
\]

(27)

\[
F_{2n}(t) = \frac{b_n}{c_n} \cos(\omega t) \left( 1 - \frac{G a_n}{s+\omega} \right) F_{n-1}(-a_n t) - \frac{G a_n}{s+\omega} F_{n-2}(-a_n t).
\]

(28)

### 3.2. Calculation of Energy

Applying the Laplace transform to Eq. (15), and Eq. (16), we obtain:

\[
s^\alpha \tilde{T}(r,s) = \frac{1}{P_e} \left[ \frac{\partial^2 \tilde{T}(r,s)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}(r,s)}{\partial r} \right] - 4 \frac{R_e}{P_e} \tilde{T}(r,s) + \frac{B_e}{P_e} \left( \frac{\omega \tilde{T}(r,s)}{s + \omega} \right)^2 + \frac{R_T}{P_e} \tilde{T}(r,s) - \tilde{T}(r,s),
\]

(29)

\[
\tilde{T}(r,s) = \frac{\gamma}{s + \gamma} \tilde{T}(r,s),
\]

(30)

Incorporating Eq. (30) into Eq. (29), we get:

\[
s^\alpha \tilde{T}(r,s) = \frac{1}{P_e} \left[ \frac{\partial^2 \tilde{T}(r,s)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}(r,s)}{\partial r} \right] - 4 \frac{R_e}{P_e} \tilde{T}(r,s) + \frac{B_e}{P_e} \left( \frac{\omega \tilde{T}(r,s)}{s + \gamma} \right)^2 + \frac{R_T}{P_e} \tilde{T}(r,s) - \tilde{T}(r,s),
\]

(31)

Applying the Hankel transform of order zero in Eq. (31), and then inserting Eq. (23) for \( w(r,s) \), we obtain:

\[
\tilde{T}_H(r,s) = \frac{J_0(r_R)}{r_R} + A_{n1} \left( \frac{b_n + \frac{b_n}{s^2 + \omega^2}}{s + \alpha_n} - \frac{\gamma - \alpha_n}{s + \alpha_n} \right),
\]

(32)

\[
\frac{\partial}{\partial r} \left[ \frac{1 - \frac{2G a_n}{s+\omega}}{c_n} \right] J_n(r_R) + \left( \frac{\gamma - \alpha_n}{s + \alpha_n} \right) A_{n1} \left( s + \alpha_n \right) \tilde{T}_H(r,s) = \frac{c_n}{P_e} \tilde{T}_H(r,s),
\]

(33)

Where:

\[
b_{n1} = \frac{b_n + \frac{b_n}{s^2 + \omega^2}}{s + \alpha_n}, \quad \alpha_{n1} = \frac{b_n + \frac{b_n}{s^2 + \omega^2}}{2}.
\]

(34)

The Bessel expansion is given by:

\[
J_n(x) = \frac{n!}{\pi x^\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \pi x^{-\frac{1}{2}}}{k! (2k+1)!} x^{2k+1},
\]

and, \( J_n(x) = 2J_n'x \), and \( J_n(x) - J_n(x) = -2J_n'x \).

(35)

(36)
Where;

\[ b_{a_2} = a_{a_1} + \alpha_{a_1}, \quad c_{a_2} = \sqrt{b_{a_2}^2 - 4\alpha_{a_1}a_{a_1}}, \quad k_{a_1} = \frac{b_{a_2} - c_{a_2}}{2}, \]

\[ k_{a_4} = \frac{b_{a_4} + c_{a_4}}{2}, \quad b_{a_5} = a_{a_3} + \alpha_{a_3}, \quad c_{a_5} = \sqrt{b_{a_5}^2 - 4\alpha_{a_3}a_{a_3}}, \]

\[ k_{a_7} = \frac{b_{a_7} - c_{a_7}}{2}, \quad k_{a_8} = \frac{b_{a_7} + c_{a_7}}{2}. \]

By applying Eq. (24) and Eq. (25) to Eqs. (37), (38), and Eq. (39), we obtain:

\[ T_4(t,r) = 1 + 2\sum_{n=1}^{\infty} \frac{J_0(r_{r_n})}{r_{r_n}} \int_{J_0(r_{r_n})} \left\{ F_{3n}(t) + F_{4n}(t) \right\}, \]

\[ T_5(t,r) = 2\sum_{n=1}^{\infty} \frac{J_0(r_{r_n})}{r_{r_n}} \int_{J_0(r_{r_n})} \left\{ F_{1n}(t) + F_{4n}(t) + F_{6n}(t) + F_{8n}(t) \right\}, \]

\[ T_6(t,r) = 2\sum_{n=1}^{\infty} \frac{J_0(r_{r_n})}{r_{r_n}} \int_{J_0(r_{r_n})} \left\{ A_{n} + F_{1n}(t) + F_{6n}(t) \right\}. \]

Where;

\[ F_{3n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})F_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}, \]

\[ F_{4n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})R_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}, \]

\[ F_{5n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})F_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}, \]

\[ F_{6n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})R_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}, \]

\[ F_{7n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})F_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}, \]

\[ F_{8n}(t) = \frac{A_{n}b_{a_1}c_{a_1}}{c_{a_1}^2} \int_{J_0(r_{r_n})} \left\{ (y - \alpha_{a_1})(1 - Ga_{a_1})R_{a_1}(-k_{a_1},t)F_{a_1}(-k_{a_1},t) \right\}. \]

4. Graphical Results

To understand the influence of various embedded parameters on the blood velocity \(w(r, t)\), particle velocity \(w_i(r, t)\), the Temperature of blood \(T(r, t)\), and the particle's temperature \(T(r, t)\), various graphs are plotted using Mathecad software. Fig. 2 is plotted to represent the behavior of fractional parameter \(\alpha\) on both the velocities of blood and particles for a small time \((t = 0.05)\) and large time \((t = 2)\). It is observed that for a shorter and larger time, the opposite behavior is noticed. The fractional parameter \(\alpha\) provides multiple integral curves when its value is \((0 < \alpha < 1)\), than the classical solution \((\alpha = 1)\).
Fig. 2. (Color online) Profiles for the velocity of blood and particles for $t = 0.05$ & $t = 2$, at different values of $\alpha$ when, $b_0 = 0.5$, $b_1 = 0.6$, Re = 3, $G = 0.8$, $R = 0.5$, $\beta = 0.4$ and $H = 1$.

Fig. 3. (Color online) Profiles for the velocity of blood and particles at time $t = 0.1$ and different values of Re when, $b_0 = 0.5$, $b_1 = 0.6$, $G = 0.8$, $R = 0.5$, $H = 1$ and $\beta = 0.4$. 
Fig. 4. (Color online) Profiles for velocity of blood and particles at time \( t = 2 \) and different values of \( \text{Re} \) when, \( b_0 = 0.5, b_1 = 0.6, G = 0.8, R = 0.5, H = 1 \) and \( \beta = 0.4 \).

Fig. 5. (Color online) Profiles for velocity of blood and particles at time \( t = 0.1 \) and different values of \( \beta \) when, \( \text{Re} = 3, b_0 = 0.5, b_1 = 0.6, G = 0.8, \) and \( H = 1 \).
These multiple solutions may be used to best fit the theoretical solutions with the real data. This shows the fractional-order model is more general than the classical order model. Figs. 3 & 4 show the influence of Re on the velocities of blood and particles. The blood and particles velocities increase with enhancing values of Re. Physically, it is true because Re is the ratio of arterial radius and viscosity of the blood. Therefore, the blood becomes thinner by increasing Re, and hence the velocity of the blood and particles increases, which is significant in decreasing the blood pressure and in the supply of heat to the tissues of the body and carrying drugs to the affected tissues. In Fig. 5 Both the blood velocity and particle velocity increase by increasing the values of $\beta$, because when the values of $\beta$ increase, the yield stress decreases and therefore the thickness of the blood decreases. Figs. 6 and 7 show the effect of the magnetic parameter or Hartmann number $H$ on both the particle and blood velocities. By increasing the values of $H$ both velocities decrease. It is true because by applying a magnetic field on a moving electrically conducting fluid, the Lorentz forces are produced due to the interaction of the magnetic field and induced currents that retard the blood flow. By regulating the intensity of the applied external magnetic field is useful to control the blood flow during magnetic therapy (for treating pain, such as the back, foot, or joint pain) and surgeries.

Figure 8 is plotted to represent the behavior of fractional order parameter $\alpha$ on both the temperature of blood and particles for small-time ($t = 0.1$) and large time ($t = 2$). The behavior of $\alpha$ on temperature of blood and particle is reported same as for the blood and particle velocities. Therefore, this may be a tool to control the blood and particle temperatures during treatment by using the suitable values of $\alpha$ for regulating blood and particles temperature. Figs. 9 and 10, represent the increase in temperature of both the blood and particle by increasing the values of Peclet number. As the Peclet number is the ratio of advection to diffusion, so by increasing the Peclet number the advection becomes dominant due to which temperature of the blood and particles increase, because advection is the transport of a substance by bulk motion and the properties of that substance are carried with it. Figs. 11, and 12, show the decrease in temperature of blood and particle by increasing the values of $R_w$. The blood and particle temperature is maximum at the center of the vessel and minimum at the walls of the vessel, which is significant to avoid effecting the tissues during
Fig. 7. (Color online) Profiles for velocity of blood and particles at time $t = 2$ and different values of $H$, when $b_0 = 0.5$, $b_1 = 0.6$, $Re = 3$, $G = 0.8$, $R = 0.5$, and $\beta = 0.4$.

Fig. 8. (Color online) Profiles for temperature of blood and particles at small and large values of time $t$ when $P_e = 5$, $R_a = 0.1$, $B_r = 0.1$ and $R_c = 5$. 


Fig. 9. (Color online) Profiles for temperature of blood and particles at different values $P_e$ at $t = 0.1$ when $R_a = 0.1$, $B_r = 0.1$ and $R_s = 5$.

Fig. 10. (Color online) Profiles for temperature of blood and particles at different values $P_e$ at $t = 2$ when $R_a = 0.1$, $B_r = 0.1$ and $R_s = 5$. 
Fig. 11. (Color online) Profiles for temperature of blood and particles at different values $R_a$ at $t = 0.1$ when $Pe = 0.5$, $B_r = 0.1$ and $R_s = 5$.

Fig. 12. (Color online) Profiles for temperature of blood and particles at different values $R_a$ at $t = 2$ when $Pe = 10$, $B_r = 0.1$ and $R_s = 5$. 
Fig. 13. (Color online) Profiles for temperature of blood and particles at different values $B_r$ at $t = 0.1$ when, $Pe = 5$, $Ra = 0.3$ and $Re = 5$.

Fig. 14. (Color online) Profiles for temperature of blood and particles at different values $B_r$ at $t = 2$ when $Pe = 5$, $Ra = 0.1$ and $Re = 5$. 
treatment by thermal therapy. Fig. 13, and 14 show the increase in the temperature of blood and particle by increasing the values of $B_r$. Physically, $B_r$ is the ratio of heat generated by viscous forces between the layers of the fluid to the heat transported by the conduction of molecules, so by increasing the values of $B_r$, the heat production by the viscous forces increases due to which the blood and particle temperature increase.

5. Concluding Remarks

- In small & narrow vessels of blood, the non-Newtonian behavior of blood (Casson fluid) is considered with heat conduction between particles and blood.
- In a cylindrical domain, Caputo’s fractional time derivative has been taken to examine the impact of different parameters on blood and particles.
- The exact solutions are found by using the Laplace & Hankel transformation jointly.
- The blood and particle velocities increase by increasing the values of $Re$ & $\beta$, which is significant in decreasing the blood pressure and in the supply of heat to the tissues of the body and carrying drugs to the affected tissues.
- The blood and particle velocities decrease for increasing the values of $H$ which is useful to control the blood flow during magnetic therapy (for treating pain, such as the back, foot, or joint pain), and surgeries.
- The temperature of blood and particles increase for increasing values of $P_a$ & $B_r$.
- By increasing the values of $Re$, decrease the blood and particle temperature. The blood and particle temperature is maximum in the center of the vessel and minimum at the vessel walls, which is significant to avoid affecting the tissues during treatment by thermal therapy.

References

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