Prediction of the Upper and Lower Bounds of Magnetic Vector Potentials in a Linear Magnetostatic Field with Uncertain-but-Bounded Parameters

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Uncertainty is ubiquitous in practical engineering and scientific research. The uncertainties in parameters can be treated as interval numbers. The prediction of upper and lower bounds of the response of a system including uncertain parameters is of immense significance in uncertainty analysis. This paper aims to evaluate the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field efficiently with uncertain-but-bounded parameters. The uncertain-but-bounded parameters are represented by interval notations. By performing Taylor series expansion on the magnetic vector potentials obtained from the equilibrium governing equation and by using the properties of interval mathematics, we can calculate the upper and lower bounds of the magnetic vector potentials of a linear magnetostatic field. In order to evaluate the accuracy and efficiency of the proposed method, two numerical examples are used. The results illustrate that the precision of the proposed method is acceptable for engineering applications, and the computation time of the proposed method is significantly less than that of the Monte Carlo simulation, which is the most widely used method related to uncertainties. The Monte Carlo simulation requires a large number of samplings, and this leads to significant runtime consumption.

Keywords : upper and lower bounds, magnetic vector potential, linear magnetostatic field, Taylor series expansion, interval parameter

1. Introduction

Magnetostatics is the branch of physics that addresses static magnetism and stationary magnetic fields. In engineering, linear magnetostatic field analysis is of immense significance for magnetic equipment design. However, all engineering analyses and design problems involve uncertainties of varying degrees. Owing to measurement inaccuracies, modelling approximations, manufacturing deviations, or external environmental changes, many types of uncertain information may arise, such as geometric dimensions, material properties, external loads, boundary conditions, etc [1]. Therefore, the concept of uncertainty plays a significant role in the calculation of the physical quantities of linear magnetostatic fields, such as magnetic field intensity, magnetic induction, magnetic flux, current, etc. Modelling the uncertain parameters as random variables is the most common approach for handling the problem of uncertainty. Under this circumstance, all the information about uncertain parameters is provided by the probability distribution functions, and the solutions or results will show statistical variations. Many papers have been published regarding the use of random analysis [5]. However, the information about the probabilistic distribution is usually unavailable or sometimes wrongly assumed, and the probabilistic results may be inaccurate or incorrect.

The probabilistic approach is not the only available method to handle uncertainty, which is not always random. Alternatively, the uncertain parameters can be treated as interval numbers with lower and upper bounds. In interval analysis, the uncertain parameter is denoted by a simple range [9]. When information about an uncertain parameter in the form of a probability function is not available, the interval analysis can be used. If the bounds on the magnitude of uncertain parameters are known, the range of the response of the uncertain system can be determined [11]. The Taylor series expansion is an effective method for mathematical analysis. In both engineering applications and scientific researches, we encounter many small changes

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in the system parameters. Taylor series expansion is an effective approach for describing the changes in the responses of a system if small changes are imposed on the parameters of the system. Such analysis will reveal the effect of parameter changes on the responses of a system. Many papers have been published regarding the use of Taylor series expansion [13]. Taylor series expansion combined with interval analysis can be used to handle uncertainty effectively.

In this study, Taylor series expansion is proposed for evaluating the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field with uncertain-but-bounded parameters. The uncertain parameters of the linear magnetostatic field are represented by interval notations. Each matrix or vector in the equilibrium governing equation is the function of the uncertain parameters. The equilibrium governing equation of the linear magneto-static field is affected by the change in each uncertain parameter. By performing Taylor series expansion on the magnetic vector potentials obtained from the equilibrium governing equation and by using the properties of interval mathematics, we can calculate the upper and lower bounds of the magnetic vector potentials. In order to evaluate the accuracy and efficiency of the proposed method, the upper and lower bounds of the magnetic vector potentials and the computation time of the proposed method are compared with those obtained using Monte Carlo simulation, which is always referred to as a reference solution when compared with other methods related to uncertainties. Monte Carlo simulation is very time consuming, but is also the most widely used method related to uncertainties.

The remainder of this paper is organized as follows. In Section 2, some basics of interval mathematics are briefly introduced. In Section 3, Taylor series expansion is employed for evaluating the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field with interval parameters. In Section 4, two numerical examples are provided to illustrate the accuracy and efficiency of the proposed method. Section 5 presents the conclusions of this study.

2. Interval Basics

Interval analysis is a new and rising branch of applied mathematics. It is a numerical approach that treats an interval as a new number [9]. Assume that R is a real number field. The closed interval denoted by [a, b] is the set of real numbers given by

$$[a,b] = \{x \in \mathrm{R} : a \le x \le b\}. \tag{1}$$

Although various other types of intervals, such as open

or half-open, appear throughout mathematics, our work will primarily concentrate on closed intervals. In this study, the term *interval* indicates a closed interval. We denote the interval as $x^I = [\underline{x}, \overline{x}]$. The left and right endpoints of the interval are denoted by $\underline{x}$ and $\overline{x}$, respectively. Moreover, $\underline{x}$ and $\overline{x}$ are the lower and upper bounds, respectively. The set of all closed real intervals is denoted by $I(R)$.

The nominal value of an interval $x^I = [\underline{x}, \overline{x}]$ is defined as:

$$x^c = m(x^I) = (\overline{x} + \underline{x})/2, \tag{2}$$

and the deviation amplitude of $x^I = [\underline{x}, \overline{x}]$ is

$$\Delta x = rad(x^I) = (\overline{x} - \underline{x})/2. \tag{3}$$

The quotient $\beta = \Delta x / x^c$ is called the uncertainty factor and it indicates the extent to which an uncertain parameter varies [10].

Further, we consider the concept of an interval vector. A vector whose elements are intervals is called an interval vector. Assume that $x_i^I = [\underline{x_i}, \overline{x_i}] \in I(\mathrm{R})$, $i = 1, 2, \ldots n$, we obtain

$$\boldsymbol{x}^I = [x_1^I, x_2^I, \cdots, x_n^I]^T, \tag{4}$$

or in the element form, we obtain

$$\boldsymbol{x}^I = ([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}], \cdots, [\underline{x_n}, \overline{x_n}])^T, \tag{5}$$

where $\boldsymbol{x}^I$ is called an *n*-dimensional interval vector. The set of all the interval vectors in R^n is denoted by $I(\mathrm{R}^n)$.

Further, the interval vector $\boldsymbol{x}^I \in I(\mathrm{R}^n)$ can also be denoted as:

$$\boldsymbol{x}^I = [\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}], \tag{6}$$

where

$$\underline{\boldsymbol{x}} = [\underline{x_1}, \underline{x_2}, \cdots, \underline{x_n}]^T, \quad \overline{\boldsymbol{x}} = [\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n}]^T. \tag{7}$$

We define the nominal value vector as:

$$\boldsymbol{x}^c = m(\boldsymbol{x}^I) = (\overline{\boldsymbol{x}} + \underline{\boldsymbol{x}})/2, \tag{8}$$

and the deviation amplitude vector as:

$$\Delta \boldsymbol{x} = rad(\boldsymbol{x}^I) = (\overline{\boldsymbol{x}} - \underline{\boldsymbol{x}})/2. \tag{9}$$

Using the interval basics in this section, we can formulate the problem of evaluating the upper and lower bounds of magnetic vector potentials in a linear magneto-static field with interval parameters.

3. Taylor Series Expansion for Evaluating the Bounds of Magnetic Vector Potentials

The finite element method (FEM) is a well-accepted

computational tool for assessing the physical properties of magnetic equipment and devices [17]. The magnetic vector potential is a fundamental parameter in magnetic FEM analysis. The equilibrium governing equation [19] for a general engineering linear magnetostatic system with n degrees of freedom is:

$$K(a) \cdot A(a) = f(a), \qquad (10)$$

where $K(a) = (k_{ij}) \in \mathrm{R}^{n \times n}$ is the stiffness matrix, $f(a) = (f_i) \in \mathrm{R}^n$ is the external excitation vector, and $A(a) = (A_i) \in \mathrm{R}^n$ is the unknown nodal vector of the magnetic vector potential. The stiffness matrix, external excitation vector, and magnetic vector potential can be regarded as functions of the system parameter vector $a = [a_1, a_2, ..., a_m]$, and m is the number of uncertain parameters.

The vector a is usually assumed to be deterministic. Considering uncertainty, the parameter vector $a = [a_1, a_2, ..., a_m]$ can be assumed to have uncertainties. Based on the interval analysis, the parameter vector $a = [a_1, a_2, ..., a_m]$ is assumed to belong to an interval vector $a^I = [a_1^I, a_2^I, ..., a_m^I]$, i.e.,

$$a \in a^I, \qquad (11)$$

or in the element form as follows

$$a_j \in a_j^I = [\underline{a}_j, \overline{a}_j], \quad j = 1, 2, \cdots, m, \qquad (12)$$

where a_j^I is an interval number, and $\underline{a}_j, \overline{a}_j$ are the lower and upper bounds of a_j^I, respectively.

Based on the interval analysis, the nominal value vector of a^I can be expressed by Eq. (8), and the deviation amplitude vector of a^I can be expressed by Eq. (9).

Using the expression of the nominal value vector a^c and deviation amplitude vector Δa, the interval parameter vector a^I can be denoted as

$$a^I = a^c + \Delta a^I, \qquad (13)$$

where Δa^I is a symmetric interval vector expressed as

$$\Delta a^I = [-\Delta a, \, \Delta a]. \qquad (14)$$

Further, the magnetic vector potential solution to Eq. (10) is an interval vector $A^I(a)$,

$$A^I(a) = \{A(a) : A(a) \in \mathrm{R}^n, \, K(a) \cdot A(a) = f(a), \, a \in a^I\}. \qquad (15)$$

The solution to Eq. (15) can be converted into the prediction of the upper and lower bounds of the magnetic vector potential, i.e.,

$$\begin{cases} \overline{A}(a) = \max\{A(a) : A(a) \in \mathrm{R}^n, \, K(a) \cdot A(a) = f(a), \, a \in a^I\} \\ \underline{A}(a) = \min\{A(a) : A(a) \in \mathrm{R}^n, \, K(a) \cdot A(a) = f(a), \, a \in a^I\} \end{cases}, \qquad (16)$$

where max and min represent the global maximum and global minimum of the set, respectively.

The upper and lower bounds of the magnetic vector potential can also be expressed in element form as follows:

$$\begin{cases} \overline{A}_j(a) = \max\{A_j(a) : a \in a^I\}, \quad j = 1, 2, \cdots, n \\ \underline{A}_j(a) = \min\{A_j(a) : a \in a^I\}, \quad j = 1, 2, \cdots, n \end{cases}, \qquad (17)$$

where n is the number of degrees of freedom in the linear magnetostatic equilibrium governing equation.

It is usually difficult to calculate the upper and lower bounds in Eq. (16) directly. Instead, Taylor series expansion can be employed to evaluate the range [11, 12]. In engineering applications, the higher-order terms in the Taylor series expansion are usually neglected, and only the first-order expansion terms are retained. When the changes in the system parameters are small, the first-order Taylor series expansion satisfies the accuracy requirement.

If the changes in the system parameters are relatively large, i.e. the deviation amplitude of the interval parameter is relatively large, the interval of the uncertain parameter can be partitioned into several subintervals, and the deviation amplitude of each subinterval is much smaller than that of the original interval [23]. For example, the interval $x^I = [1.8, 2.2]$ can be divided into two subintervals, i.e., $x_1^I = [1.8, 2.0]$ and $x_2^I = [2.0, 2.2]$, and $x^I = x_1^I \cup x_2^I$. The deviation amplitude of the original interval is $\Delta x = 0.2$, and the deviation amplitude of the subinterval is $\Delta x_1 = \Delta x_2 = 0.1$. Once an interval is divided into several subintervals, the deviation amplitude becomes significantly smaller, and in this situation, the first-order Taylor series expansion can be directly used to predict the upper and lower bounds of the response to a system with uncertainties. Thus, we study only the case in which the deviation amplitude of the interval parameter is small.

For an uncertain variable x, the quotient $\beta = \Delta x / x^c (x^c \neq 0)$ is the uncertainty factor. In this study, the uncertainty factors of the uncertain parameters are assumed to be smaller than 5 %, i.e. $\beta \leq 5 \%$. In practice, uncertainty of 5 % is sufficiently large for most engineering parameters, and hence, this is an acceptable range for applications. Thus, the changes in the parameters are treated as small changes in this study.

By performing the first-order Taylor series expansion of $A(a)$ about $a = a^c$, we obtain

$$A(a) = A(a^c) + \sum_{j=1}^{m} \frac{\partial A(a^c)}{\partial a_j} \delta a_j, \qquad (18)$$

and

$$\delta a_j \in \Delta a_j^I = [-\Delta a_j,\ \Delta a_j],\ j=1,2,\cdots,m, \tag{19}$$

where m is the number of uncertain parameters.

Substituting Eq. (19) into Eq. (18) yields

$$A^I(\boldsymbol{a}) = A(\boldsymbol{a}^c) + \sum_{j=1}^{m} \frac{\partial A(\boldsymbol{a}^c)}{\partial a_j} \Delta a_j. \tag{20}$$

By using the interval operations, the upper and lower bounds of the magnetic vector potential based on the Taylor series expansion can be determined as follows:

$$\begin{cases} \overline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) + \sum_{j=1}^{m} \left| \frac{\partial A(\boldsymbol{a}^c)}{\partial a_j} \right| \Delta a_j \\ \underline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) - \sum_{j=1}^{m} \left| \frac{\partial A(\boldsymbol{a}^c)}{\partial a_j} \right| \Delta a_j \end{cases}. \tag{21}$$

Substituting $\boldsymbol{a} = \boldsymbol{a}^c$ into Eq. (10) yields

$$\boldsymbol{K}(\boldsymbol{a}^c) \cdot A(\boldsymbol{a}^c) = \boldsymbol{f}(\boldsymbol{a}^c). \tag{22}$$

The partial derivative $\dfrac{\partial A(\boldsymbol{a}^c)}{\partial a_j}$ can be derived as follows [27-30]. The stiffness matrix $\boldsymbol{K}(\boldsymbol{a})$ and the external excitation vector $\boldsymbol{f}(\boldsymbol{a})$ are assumed to be differentiable with respect to each parameter $a_j, (j=1,2,\cdots,m)$, and this guarantees the differentiability of $A(\boldsymbol{a})$ with respect to each parameter $a_j, (j=1,2,\cdots,m)$.

Differentiating both sides of Eq. (22) with respect to a_j yields

$$\frac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \cdot A(\boldsymbol{a}^c) + \boldsymbol{K}(\boldsymbol{a}^c) \cdot \frac{\partial A(\boldsymbol{a}^c)}{\partial a_j} = \frac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j},\ j=1,2,\cdots,m. \tag{23}$$

By transposing the terms, the partial derivative $\dfrac{\partial A(\boldsymbol{a}^c)}{\partial a_j}$ can be obtained as follows:

$$\frac{\partial A(\boldsymbol{a}^c)}{\partial a_j} = \boldsymbol{K}^{-1}(\boldsymbol{a}^c) \cdot \left(\frac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \cdot A(\boldsymbol{a}^c) \right),\ j=1,2,\cdots,m. \tag{24}$$

By substituting Eq. (24) into Eq. (21), and using the partial derivatives derived from Eq. (26), the upper and lower bounds of the magnetic vector potential can be calculated as follows:

$$\begin{cases} \overline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) + \sum_{j=1}^{m} \left| \boldsymbol{K}^{-1}(\boldsymbol{a}^c) \cdot \left(\frac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \cdot A(\boldsymbol{a}^c) \right) \right| \Delta a_j \\ \underline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) - \sum_{j=1}^{m} \left| \boldsymbol{K}^{-1}(\boldsymbol{a}^c) \cdot \left(\frac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j} - \frac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \cdot A(\boldsymbol{a}^c) \right) \right| \Delta a_j \end{cases}. \tag{25}$$

Notably, the matrix and vector partial derivatives, $\dfrac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j}$ and $\dfrac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j}$ in Eq. (24) are still unknown. It is usually very difficult to obtain these partial derivatives using an analytical approach. An alternative feasible approach for engineering applications is to perturb one system parameter at a time and use the finite difference method to approximate the partial derivatives. It is illustrated as follows:

$$\begin{cases} \dfrac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \approx \dfrac{\boldsymbol{K}(a_1^c, a_2^c, \cdots, a_j^c + \varepsilon_j, \cdots, a_m^c) - \boldsymbol{K}(a_1^c, a_2^c, \cdots, a_j^c, \cdots, a_m^c)}{\varepsilon_j} \\ \dfrac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j} \approx \dfrac{\boldsymbol{f}(a_1^c, a_2^c, \cdots, a_j^c + \varepsilon_j, \cdots, a_m^c) - \boldsymbol{f}(a_1^c, a_2^c, \cdots, a_j^c, \cdots, a_m^c)}{\varepsilon_j} \end{cases}, \tag{26}$$

where ε_j is a small perturbation of the system parameter $a_j, (1 \le j \le m)$. For example, if $a_j^c = 1$, the small perturbation ε_j can be considered as $\varepsilon_j = 0.005$.

For the sake of simplicity, we consider the notation as follows:

$$\boldsymbol{a}^c + \varepsilon_j = [a_1^c, a_2^c, \cdots, a_j^c + \varepsilon_j, \cdots, a_m^c], \tag{27}$$

and Eq. (26) can be simplified as:

$$\begin{cases} \dfrac{\partial \boldsymbol{K}(\boldsymbol{a}^c)}{\partial a_j} \approx \dfrac{\boldsymbol{K}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{K}(\boldsymbol{a}^c)}{\varepsilon_j} \\ \dfrac{\partial \boldsymbol{f}(\boldsymbol{a}^c)}{\partial a_j} \approx \dfrac{\boldsymbol{f}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{f}(\boldsymbol{a}^c)}{\varepsilon_j} \end{cases}. \tag{28}$$

By substituting Eq. (28) into Eq. (25), the upper and lower bounds of the magnetic vector potential can be calculated as follows:

$$\begin{cases} \overline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) \\ \quad + \sum_{j=1}^{m} \left| \boldsymbol{K}^{-1}(\boldsymbol{a}^c) \cdot \left(\dfrac{\boldsymbol{f}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{f}(\boldsymbol{a}^c)}{\varepsilon_j} - \dfrac{\boldsymbol{K}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{K}(\boldsymbol{a}^c)}{\varepsilon_j} \cdot A(\boldsymbol{a}^c) \right) \right| \Delta a_j \\ \underline{A}(\boldsymbol{a}) = A(\boldsymbol{a}^c) \\ \quad - \sum_{j=1}^{m} \left| \boldsymbol{K}^{-1}(\boldsymbol{a}^c) \cdot \left(\dfrac{\boldsymbol{f}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{f}(\boldsymbol{a}^c)}{\varepsilon_j} - \dfrac{\boldsymbol{K}(\boldsymbol{a}^c + \varepsilon_j) - \boldsymbol{K}(\boldsymbol{a}^c)}{\varepsilon_j} \cdot A(\boldsymbol{a}^c) \right) \right| \Delta a_j \end{cases}. \tag{29}$$

In practice, the stiffness matrix and external excitation vector can be obtained using the finite element software ANSYS. We can establish the finite element model in ANSYS, and the system parameters are assigned their nominal values. The stiffness matrix $\boldsymbol{K}(\boldsymbol{a}^c)$, and the external excitation vector $\boldsymbol{f}(\boldsymbol{a}^c)$ can be obtained using ANSYS. We perturb the system parameter $\varepsilon_j (1 \le j \le m)$

of the finite element model with a small perturbation $\varepsilon_j (1 \le j \le m)$. Subsequently, the perturbed stiffness matrix $K(a^c + \varepsilon_j)$ and the perturbed external excitation vector $f(a^c + \varepsilon_j)$ can also be obtained using ANSYS. All the matrices and vectors obtained from ANSYS can be loaded to MATLAB. Using the matrix operations, we can execute the calculation of Eq. (29) in MATLAB, such that the upper and lower bounds of the magnetic vector potential can be obtained.

4. Numerical Examples

In order to demonstrate the feasibility of Taylor series expansion for evaluating the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field with uncertain-but-bounded parameters, the following two numerical examples are used. The results are compared with those obtained using a Monte Carlo simulation.

Monte Carlo simulation is a statistical method that obtains the distribution range of responses through a large number of random samplings. The Monte Carlo simulation is most suitable for practical engineering applications owing to their reliance on the repeated computation of random numbers [25]. It is generally used when the computation of an exact result using a deterministic algorithm is unfeasible or impossible. It is always referred to as a reference solution when compared with other methods related to uncertainties.

The Monte Carlo simulation for predicting the upper and lower bounds of the response to a system with uncertainties always requires a large number of samplings, and this leads to significant runtime consumption. However, the proposed Taylor series expansion method related to uncertainties is significantly more time-saving because it does not demand a large sampling scale. In the following examples, the computation time of Taylor series expansion and Monte Carlo simulation are compared.

4.1. Example 1

Consider the linear magnetostatic field distribution of two current-carrying conductors in a two-dimensional space, as shown in Fig. 1. The linear magnetostatic field of two current-carrying conductors.. The electric current flows through the two conductors in the same direction, along the z-axis. The two conductors are assumed to be infinitely long in the z-direction; hence, the linear magnetostatic field distribution can be regarded as a two-dimensional problem in the xoy-plane. The thickness of each conductor is $t = 2$ mm, the width of each conductor in the y-direction is $a = 12$ mm, and the distance between the center points of the two conductors in the x-direction
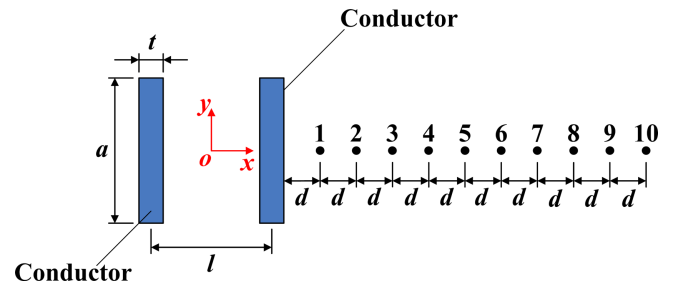

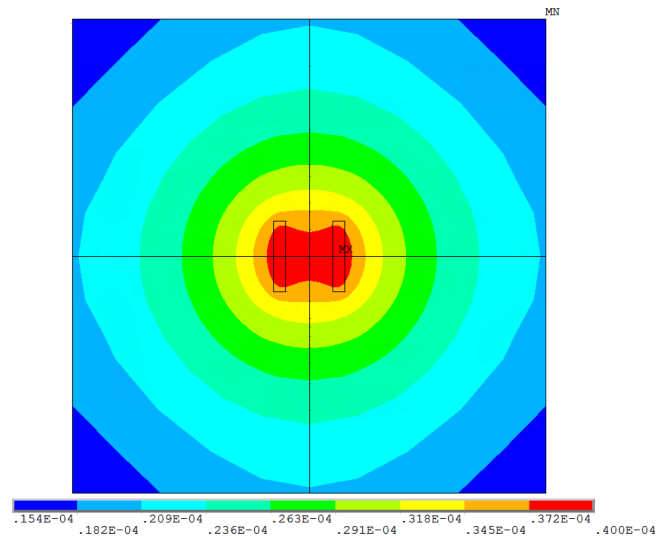**Fig. 1.** (Color online) The linear magnetostatic field of two current-carrying conductors.

Fig. 2. (Color online) The planar contour of the magnetic vector potentials in the z-direction with nominally valued parameters in Example 1.

is $l = 10$ mm. The two conductors are surrounded by air. Points 1-10 are located on the x-axis in the xoy-plane, and the distance between every two points is $d = 3$ mm. The linear magnetostatic field is symmetrical about the yoz-plane. The magnetic vector potentials of points 1-10 are discussed in the following text. The magnetic relative permeability of air is $\mu_0 = 1$, and the magnetic relative permeability of the conductor is $\mu = 1$. The current applied to each conductor in the z-direction is $I = 24$ A. When all parameters are assigned their nominal values, the planar contour of the magnetic vector potentials in the z-direction is plotted in Fig. 2.

Owing to manufacturing inaccuracy, measurement errors, or variances of the environment, the parameters exhibit some uncertainties. In this example, the magnetic relative permeability of the conductor and the magnitude of current are assumed to be the interval parameters. The magnetic relative permeability of the conductor μ is assumed to vary within the interval $\mu^I = [0.95, 1.05]$; i.e.,

the nominal value is $\mu^c = 1$, and the deviation amplitude is $\Delta\mu = 0.05$. The current applied to each conductor is assumed to vary within the interval $I^I = [22.8, 25.2]$; i.e., the nominal value is $I^c = 24$, and the deviation amplitude is $\Delta I = 1.2$.

The upper and lower bounds of magnetic vector potentials of points 1-10 are discussed as follows. The distribution of the linear magnetostatic field is a two-dimensional problem, and the magnetic vector potentials contain only components in the y-direction. Using the given interval information, a Taylor series expansion is employed to predict the upper and lower bounds of the magnetic vector potentials. In order to verify the accuracy and efficiency of the proposed method, a Monte Carlo simulation is performed for comparison. In this example, the sampling size of Monte Carlo simulation is 10^6, which is large enough for most engineering applications. The sample data are generated using the optimal Latin hyper-cube design method, which distributes the sample data over the design variable space in a uniform manner and can avoid the occurrence of data clustering. This design method is commonly used in sample data generation.

The upper and lower bounds obtained by using the

Table 1. Nominal values and upper bounds of the magnetic vector potentials (10^{-5} Wb/m) in Example 1.

Point No.	Nominal value $A_{z,i}^c$	$\overline{A}_{z,i}$ by Taylor series expansion	$\overline{A}_{z,i}$ by Monte Carlo simulation	Relative error
1	3.4942	3.6694	3.6651	0.117 %
2	3.2156	3.3766	3.3727	0.115 %
3	3.0053	3.1557	3.1520	0.117 %
4	2.8334	2.9751	2.9717	0.114 %
5	2.6885	2.8230	2.8197	0.117 %
6	2.5638	2.6921	2.6890	0.115 %
7	2.4549	2.5776	2.5747	0.112 %
8	2.3586	2.4766	2.4737	0.117 %
9	2.2732	2.3869	2.3841	0.117 %
10	2.1968	2.3066	2.3040	0.117 %

Table 2. Nominal values and lower bounds of the magnetic vector potentials (10^{-5} Wb/m) in Example 1.

Point No.	Nominal value $A_{z,i}^c$	$\underline{A}_{z,i}$ by Taylor series expansion	$\underline{A}_{z,i}$ by Monte Carlo simulation	Relative error
1	3.4942	0.3319	0.3332	0.390 %
2	3.2156	0.3055	0.3067	0.391 %
3	3.0053	0.2855	0.2866	0.383 %
4	2.8334	0.2692	0.2702	0.371 %
5	2.6885	0.2554	0.2564	0.390 %
6	2.5638	0.2436	0.2445	0.368 %
7	2.4549	0.2332	0.2341	0.384 %
8	2.3586	0.2241	0.2249	0.356 %
9	2.2732	0.2160	0.2168	0.369 %
10	2.1968	0.2087	0.2095	0.382 %

Monte Carlo simulation are considered as control group results. In this example, $A_{z,i}^c$ denotes the nominal value, and $\overline{A}_{z,i}$ and $\underline{A}_{z,i}$ denote the upper and lower bounds of the magnetic vector potential of the ith point in the z-direction, respectively. The nominal values and the upper bounds of the magnetic vector potentials obtained by using the Taylor series expansion and Monte Carlo simulation are listed in Table 1. The nominal values and the lower bounds of the magnetic vector potentials obtained by using the Taylor series expansion and Monte Carlo simulation are listed in Table 2. The relative errors of the upper and lower bounds are also calculated.

The testing environments and computation time of Taylor series expansion method and Monte Carlo simulation are compared in Table 3. It can be observed that with the hardware environment remaining the same, the computation time of Taylor series expansion method is approximately 2 min, whereas, the computation time of Monte Carlo simulation is more than 5 h. Obviously, the proposed method is significantly more time-saving.

The results presented in Table 1 and Table 2 indicate that the Taylor series expansion results are consistent with those obtained using Monte Carlo simulation. The relative

Table 3. Comparison of hardware environment and computation time in Example 1.

Item	Taylor series expansion	Monte Carlo simulation
Operating system :	Windows 7	Windows 7
CPU :	Intel i7-4770@3.40 GHz	Intel i7-4770@3.40 GHz
System type :	64 bit	64 bit
Memory :	16 GB	16 GB
Number of processors :	8	8
Software used :	ANSYS15.0, MATLAB R2013	ANSYS 15.0
Sampling size :	—	10^6
Computation time :	135.72 s (2.26 min)	21258.94 s (5.91 h)

errors are within acceptable ranges. The results presented in Table 3 indicate that, with the hardware environment remaining the same, the computation time of Taylor series expansion method is significantly less than that of Monte Carlo simulation. The results of comparison demonstrate the accuracy and efficiency of the application of Taylor series expansion for evaluating the upper and lower bounds of magnetic vector potentials in a linear magneto-static field with interval parameters.

4.2. Example 2

Consider the linear magnetostatic field distribution of a cylindrical solenoid brake system as shown in Fig. 3. The system is symmetrical about its central vertical axis, and the plot in Fig. 3. The linear magnetostatic field of a cylindrical solenoid brake system. is half the sectional view. The symmetry axis is located in the cutting plane of the sectional view. The solenoid brake system consists of three parts: the armature part, the coil part, and the back iron part. The armature part is the motion part of the system and is located at the top. The back iron part is the stationary part of the system and is located at the bottom. Between the armature part and the back iron part is the coil part, which comprises 650 turns of copper wires. There is electric current flowing through the copper wires. It is assumed that the magnetic flux generated by the coil current is very small and that there is no magnetic

saturation of the system. Thus, the linear magnetostatic field analysis of the system is linear and requires only one step of iteration to obtain the solutions to the unknown magnetic parameters. To simplify the analysis, it is also assumed that there is no magnetic leakage at the boundary parts of the system. There is air in the gap between the components.

Because the system is symmetrical about its central vertical axis, the linear magnetostatic field distribution can be regarded as a two-dimensional problem on the xoy-plane, where the y-axis is the symmetry axis. The dimensions of the back iron part are as follows: $t_1 = 7.5$ mm, $t_2 = 7.5$ mm, and $t_3 = 5.0$ mm. The thickness of the armature part is $t_4 = 7.5$ mm. The gap between the armature part and the back iron part is $t_0 = 2.5$ mm. The space between the coil part and the back iron part is $d = 2.5$ mm. The width of the coil part is $W = 10$ mm, and the height of the coil part is $H = 20$ mm. Points 1-6 are six representative locations of the system, and their magnetic vector potentials are discussed in the following. The mag-netic relative permeability of air is $\mu_{air} = 1$, the magnetic relative permeability of the copper wire in the coil part is $\mu_{coil} = 1$, the magnetic relative permeability of the back iron is $\mu_{back\ iron} = 1000$, and the magnetic relative perme-ability of the armature is $\mu_{armature} = 2000$. The current applied to each copper wire of the coil part is $I = 1$ A. The number of copper wire turns on the coil part is $N = 650$. When all parameters are assigned their nominal values, the three-dimensional contour of the magnetic vector potentials is plotted in Fig. 4.

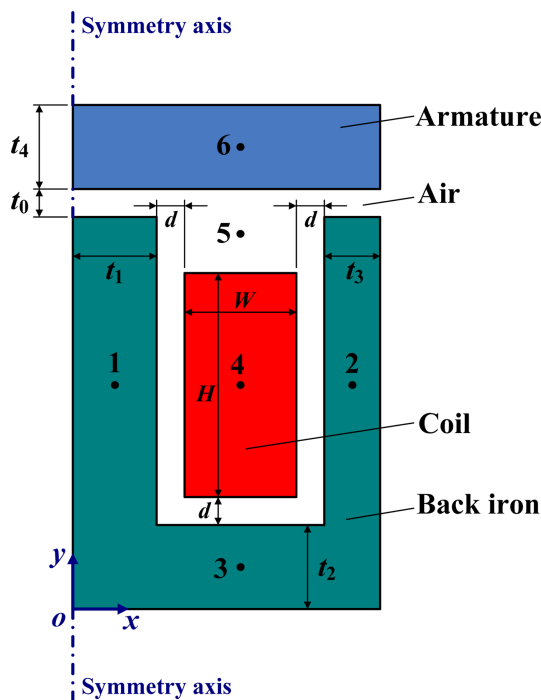Owing to manufacturing inaccuracy, measurement errors,

Fig. 3. (Color online) The linear magnetostatic field of a cylin-drical solenoid brake system.
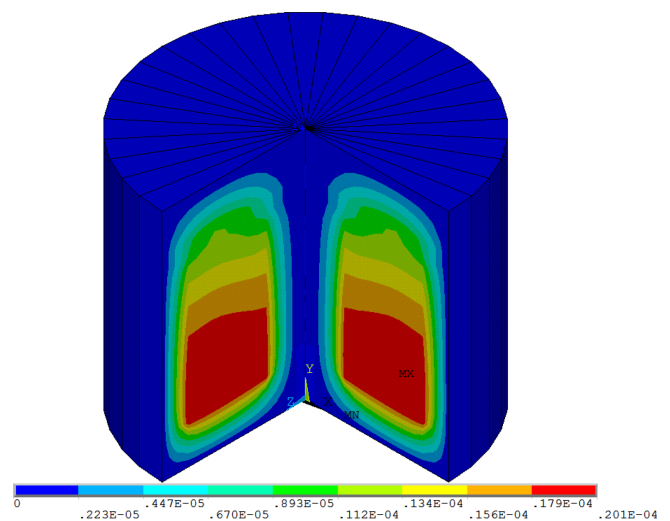

Fig. 4. (Color online) The three-dimensional contour of the magnetic vector potentials with nominally valued parameters in Example 2.

or variances of the environment, the parameters exhibit some uncertainties. In this example, the magnetic relative permeabilities of the back iron and the armature, and the magnitude of the current applied to each copper wire of the coil part, are assumed to be the interval parameters. The magnetic relative permeability of the back iron $\mu_{back\ iron}$ = 1 is assumed to vary within the interval $\mu_{back\ iron}^I$ = [950, 1050]; i.e., the nominal value is $\mu_{back\ iron}^I$ = 1000, and the deviation amplitude is $\Delta\mu_{back\ iron}^I$ = 50. The magnetic relative permeability of the armature $\mu_{armature}$ is assumed to vary within the interval $\mu_{armature}^I$ = [1900, 2100]; i.e., the nominal value is $\mu_{armature}^c$ = 2000, and the deviation amplitude is $\Delta\mu_{armature}$ = 100. The current applied to each copper wire of the coil part is assumed to vary within the interval I^I = [0.95, 1.05]; i.e., the nominal value is I^c = 1, and the deviation amplitude is ΔI = 0.05.

The upper and lower bounds of magnetic vector potentials of points 1-6 are discussed as follows. The distribution of the linear magnetostatic field in Fig. 3 is a two-dimensional axisymmetric problem. Using the given interval information, a Taylor series expansion is employed to predict the upper and lower bounds of the magnetic vector potentials. In order to verify the accuracy and efficiency of the proposed method, a Monte Carlo simulation is performed for comparison. In this example, the sampling size of Monte Carlo simulation is 10^6, which is large enough for most engineering applications. The sample data are generated using the optimal Latin hypercube design method, which distributes the sample data over the design variable space in a uniform manner and can avoid the occurrence of data clustering. This design method is commonly used in sample data generation.

The upper and lower bounds obtained by using the Monte Carlo simulation are considered as control group results. In this example, $A_{z,i}^c$ denotes the nominal value, and $\overline{A}_{z,i}$ and $\underline{A}_{z,i}$ denote the upper and lower bounds of the magnetic vector potential of the ith point in the z-direction, respectively. The nominal values and the upper bounds of the magnetic vector potentials obtained by

Table 4. Nominal values and upper bounds of the magnetic vector potentials (10^{-5} Wb/m) in Example 2.

Point No.	Nominal value $A_{z,i}^c$	$\overline{A}_{z,i}$ by Taylor series expansion	$\overline{A}_{z,i}$ by Monte Carlo simulation	Relative error
1	1.2008	1.2628	1.2566	0.493 %
2	0.3876	0.4076	0.4056	0.493 %
3	0.4930	0.5185	0.5160	0.484 %
4	1.3528	1.4225	1.4156	0.487 %
5	0.8123	0.8541	0.8501	0.470 %
6	0.3901	0.4102	0.4082	0.490 %

Table 5. Nominal values and lower bounds of the magnetic vector potentials (10^{-5} Wb/m) in Example 2.

Point No.	Nominal value $A_{z,i}^c$	$\underline{A}_{z,i}$ by Taylor series expansion	$\underline{A}_{z,i}$ by Monte Carlo simulation	Relative error
1	1.2008	1.1388	1.1432	0.385 %
2	0.3876	0.3676	0.3690	0.379 %
3	0.4930	0.4676	0.4694	0.383 %
4	1.3528	1.2831	1.2880	0.380 %
5	0.8123	0.7705	0.7734	0.375 %
6	0.3901	0.3701	0.3714	0.350 %

using the Taylor series expansion and Monte Carlo simulation are listed in Table 4. The nominal values and the lower bounds of the magnetic vector potentials obtained by using the Taylor series expansion and Monte Carlo simulation are listed in Table 5. The relative errors of the upper and lower bounds are also calculated.

The testing environments and computation time of Taylor series expansion method and Monte Carlo simulation are compared in Table 6. It can be observed that with the hardware environment remaining the same, the computation time of Taylor series expansion method is approximately 11 min, whereas, the computation time of Monte Carlo simulation is more than 46 h. Obviously, the proposed method is significantly more time-saving.

Table 6. Comparison of hardware environment and computation time in Example 2.

Item	Taylor series expansion	Monte Carlo simulation
Operating system :	Windows 7	Windows 7
CPU :	Intel i7-4770@3.40 GHz	Intel i7-4770@3.40 GHz
System type :	64 bit	64 bit
Memory :	16 GB	16 GB
Number of processors :	8	8
Software used :	ANSYS15.0, MATLAB R2013	ANSYS 15.0
Sampling size :	—	10^6
Computation time :	685.18 s (11. 42 min)	169043.32 s (46.95 h)

The results presented in Table 4 and Table 5 indicate that the Taylor series expansion results are consistent with those obtained using Monte Carlo simulation. The relative errors are within acceptable ranges. The results presented in Table 6 indicate that, with the hardware environment remaining the same, the computation time of Taylor series expansion method is significantly less than that of Monte Carlo simulation. The results of comparison demonstrate the accuracy and efficiency of the application of Taylor series expansion for evaluating the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field with interval parameters.

5. Conclusions

The Taylor series expansion is proposed for evaluating the upper and lower bounds of magnetic vector potentials in a linear magnetostatic field with uncertain-but-bounded parameters by using the interval notations. Numerical examples illustrate that the bounds of magnetic vector potentials of this method agree well with those obtained using a Monte Carlo simulation. Moreover, the proposed method is significantly more time-saving than Monte Carlo simulation. The results of the study indicate that the Taylor series expansion can estimate the upper and lower bounds of magnetic vector potentials in a linear magneto-static field with interval parameters efficiently.

Uncertainties are ubiquitous in practical magnetostatic engineering: the dimensions of a magnetostatic system may vary owing to measurement inaccuracies or manu-facturing deviations, material properties may vary owing to changes in humidity or temperature, applied currents may vary owing to instability of the power equipment, magnetic relative permeabilities may vary owing to improper contact with the adjacent conductor materials, and the experimental data may vary owing to different observations by different laboratory staff. In such situations, the proposed method of introducing Taylor series expansion for pre-dicting the upper and lower bounds of magnetic vector potentials can be employed. Our proposed method possesses broad prospects in magnetostatic engineering applications.

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