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A Comparative Study on Estimation Methods for Statistical Moments of Electromagnetic Performance Functions

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A performance moment integration method is proposed to accurately estimate the first two statistical moments, mean and variance, of electromagnetic performance functions in the present of uncertainties. To maximize computational efficiency, its numerical integration is executed not on the input design domain but on the output performance domain, where quadrature points are explored by the first order reliability analysis method. For better understating between statistical moment analysis methods, two different numerical integration schemes of dimension reduction method and performance moment integration method are compared with each other. Finally, a mathematical model and a loudspeaker design model are tested to demonstrate the features of two methods and to examine their numerical accuracy and efficiency.

Keywords : electromagnetics, optimization, reliability theory, robustness

1. Introduction

Incorporation of uncertainties into an early design stage has become increasingly important to produce robust electromagnetic (EM) devices or systems. In robust design, the product quality can be described by use of the first two statistical moments of an EM performance function: mean and variance. Therefore, it is necessary to develop accurate and efficient numerical methods, which can estimate the two statistical moments of the performance function and their sensitivities. In accordance with this demand, various attempts such as worst-case scenario, experimental design technique, Monte Carlo Simulation (MCS), dimension reduction method (DRM) and so on have been made in our community to date [1-9]. They all focus on improving the product quality through minimizing variability of the output performance function.

Among them, it has been revealed that only two numerical techniques of MCS and DRM can quantitatively assess the first two statistical moments of EM-related performance functions [4-7]. However, both of them have individual merits and demerits when applied to real-world engineering problems. The MCS could be accurate for the moment estimation, but it requires a huge number of function evaluations, which is apt to cause an unacceptable computation cost [5]. To alleviate the above difficulty, the univariate DRM in [7] was introduced to evaluate the statistical moments and their sensitivities of EM performance functions. Therein any n-dimensional function is additively decomposed into one-dimension ones, and then moment-based numerical integration is conducted on the input design domain (i.e. system input). The method can yield satisfactory results on relatively low dimensional functions, but its computational cost rapidly increases as the dimension is getting higher.

To overcome the difficulty in DRM, this paper proposes the performance moment integration (PMI) method, which can directly identify uncertainty propagation in EM performance functions. Unlike the DRM, the statistical moments of the performance function are evaluated through a numerical integration on the output performance domain (i.e. system output). The method basically makes use of three-level numerical integration scheme, and accordingly three quadrature points are sought out by the first order reliability analysis on the output domain. Through testing a mathematical example and a loudspeaker design problem, numerical accuracy and efficiency of the proposed method is examined by comparison with the existing univariate DRM.

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2. Evaluation of Statistical Moments and Their Sensitivities

The main concern of quality management for EM products is how accurately and efficiently the statistical moments and their sensitivities of the performance function $h(\mathbf{x})$ can be estimated. The kth statistical moment of $h(\mathbf{x})$ is analytically obtained from the following integration:

$$E(\{h(\mathbf{x})\}^k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{h(\mathbf{x})\}^k f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (1)$$

where $f_{\mathbf{x}}(\mathbf{x})$ is a joint probability density function (PDF) of a design random variable vector $\mathbf{x}$. Especially when the dimension of $h(\mathbf{x})$ is relatively high, it is practically impossible to calculate the multiple integration of (1). To tackle this difficulty, two different integration schemes are explained.

2.1. Univariate DRM on Input Domain

The univariate DRM additively decomposes any N-dimensional performance function into one-dimension ones on the input design domain [5, 7]:

$$h(\mathbf{x}) \cong \sum_{i=1}^{N} h(\mu_1, \cdots, \mu_{i-1}, x_i, \mu_{i+1}, \cdots, \mu_N) - (N-1) h(\mu_1, \cdots, \mu_N) \quad (2)$$

where μ_i is the mean value of the ith random variable x_i, and N is the number of random variables. The one-dimensional numerical integration can be computed by using the moment-based integration rule (MBIR) [5], which is similar to Gaussian quadrature. According to the rule, the kth statistical moment of a one-dimensional function is written by

$$E(\{h(\mathbf{x})\}^k) \cong \sum_{j=1}^{n} w^j h^k(x^j) \quad (3)$$

where w^j is the weight factor, x^j is the quadrature point, and n is the number of weights and quadrature points. Providing that the PDF of random variables is given, the weights and quadrature points comply with MBIR. For random variables with standard normal probabilistic distributions, three quadrature points and weights are presented in Table 1.

Table 1. Three Quadrature Points and Weights for a Standard Normal PDF.

Quadrature points			Weights		
x^1	x^2	x^3	w^1	w^2	w^3
$-\sqrt{3}$	0	$\sqrt{3}$	1/6	4/6	1/6

Using (2) and (3), the mean m_h and variance σ_h^2 of $h(\mathbf{x})$ is expressed as

$$\begin{aligned} \mu_h &\equiv E(h(\mathbf{x})) \\ &\cong \sum_{i=1}^{N} \sum_{j=1}^{n} w_i^j h(\mu_1, \cdots, x_i^j, \cdots, \mu_N) - (N-1) h(\mu_1, \cdots, \mu_N) \end{aligned} \quad (4)$$

$$\begin{aligned} \sigma_h^2 &\equiv E(\{h(\mathbf{x}) - \mu_h\}^2) \\ &\cong \sum_{i=1}^{N} \sum_{j=1}^{n} w_i^j h^2(\mu_1, \cdots, x_i^j, \cdots, \mu_N) - (N-1) h^2(\mu_1, \cdots, \mu_N) - \mu_h^2 \end{aligned} \quad (5)$$

where w_i^j and x_i^j mean the jth weight factor and quadrature point for the ith random variable x_i, respectively. In case of implementing robust design optimization (RDO), not only t the first two statistical moments but also their sensitivities are needed [5], [7]. By applying the partial derivative to (4) and (5), the sensitivities of two statistical moments at the kth design variable ($d_k = \mu_k$) are derived as follows:

$$\partial \mu_h / \partial \mu_k \cong \sum_{i=1}^{N} \sum_{j=1}^{n} w_i^j (\partial h / \partial x_k)|_{\mathbf{x}=(\mu_1, \cdots, x_i^j, \cdots, \mu_N)} - (N-1)(\partial h / \partial x_k)|_{\mathbf{x}=\boldsymbol{\mu}} \quad (7)$$

$$\begin{aligned} \partial \sigma_h^2 / \partial \mu_k \cong & \sum_{i=1}^{N} \sum_{j=1}^{n} w_i^j (\partial h^2 / \partial x_k)|_{\mathbf{x}=(\mu_1, \cdots, x_i^j, \cdots, \mu_N)} \\ & - (N-1)(\partial h^2 / \partial x_k)|_{\mathbf{x}=\boldsymbol{\mu}} - \partial \mu_h^2 / \partial x_k . \end{aligned} \quad (8)$$

After all, the sensitivities of statistical moments can be approximated with the derivative of decomposed one-dimensional functions.

2.2. PMI on Output Domain

The random variable vector $\mathbf{x}$ can be transformed to the standard normal random variable vector $\mathbf{u}$. That is, $h(\mathbf{x})$ in X-space is mapped onto $h(T(\mathbf{x})) \equiv h(\mathbf{u})$ in U-space, where the joint PDF $f_{\mathbf{x}}(\mathbf{x})$ is equal to $\phi_{\mathbf{u}}(\mathbf{u})$. Accordingly, the multi-dimensional integral of (1) can be rewritten in terms of the output distributions on the output performance domain as

$$E(\{h(\mathbf{x})\}^k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{h(\mathbf{u})\}^k \phi_{\mathbf{u}}(\mathbf{u}) d\mathbf{u} = \int_{-\infty}^{\infty} h^k f_h(h) dh \quad (9)$$

where $f_h(h)$ is a joint PDF of $h(\mathbf{u})$. Similar to the univariate DRM, PMI also makes use of three quadrature points and weights to approximate the one-dimensional integration in (9). Using the three-level numerical integration in Table 1, the mean and variance of the output performance is discretized by

$$\mu_h \cong \frac{1}{6} h_{\beta=-\sqrt{3}} + \frac{4}{6} h(\boldsymbol{\mu}_{\mathbf{x}}) + \frac{1}{6} h_{\beta=+\sqrt{3}} \quad (10)$$

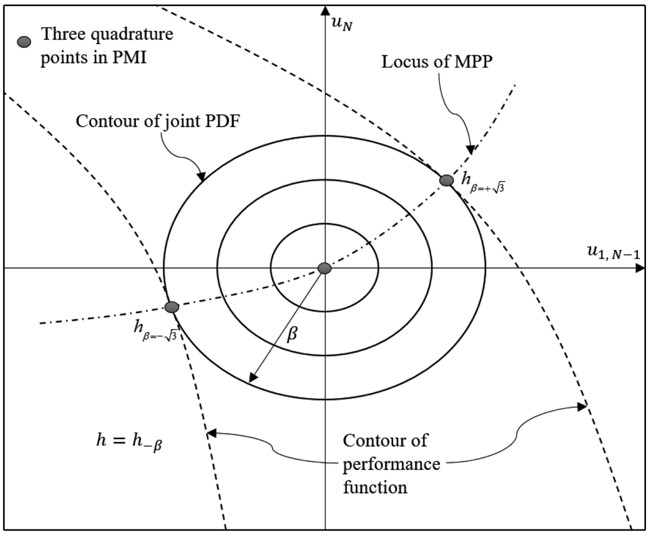

Fig. 1. Illustration of three quadrature points and MPP locus.

$$\sigma_h^2 \cong {}^1\!/_6\, h_{\beta=-\sqrt{3}}^2 + {}^4\!/_6\, h^2(\boldsymbol{\mu}_\mathbf{x}) + {}^1\!/_6\, h_{\beta=+\sqrt{3}}^2 - \mu_h^2 \quad (11)$$

where $h_{\beta=-\sqrt{3}}$ and $h_{\beta=+\sqrt{3}}$ mean performance function values at two most probable failure points (MPP) obtained through reliability analyses at a confidence level of $\beta=\sqrt{3}$ as illustrated in Fig. 1. Therein, the origin in *U*-space is corresponding to the mean vector $\boldsymbol{\mu}_\mathbf{x}$ of $\mathbf{x}$.

The function values at MPPs in *U*-space can be defined using the inverse reliability analysis called performance measure approach (PMA) like (2) as in [8].

$$\begin{aligned} &\text{maximize} && h(\mathbf{u}) \\ &\text{subject to} && \|\mathbf{u}\| = \beta = \sqrt{3}. \end{aligned} \quad (12)$$

The solution of (12) is denoted as $h_{\beta=+\sqrt{3}}$, and also the result obtained by minimizing $h(\mathbf{u})$ in (12) is denoted as $h_{\beta=-\sqrt{3}}$. Similar to the sensitivity calculation in DRM, the sensitivities of the mean and variance of the performance function with respect to the design variable m_k are easily derived as follows.

$$\frac{\partial \mu_h}{\partial \mu_k} \cong \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\mathbf{x}_{MPP}^-} + \frac{4}{6}\frac{\partial h(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\boldsymbol{\mu}_\mathbf{x}} + \frac{1}{6}\frac{\partial h(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\mathbf{x}_{MPP}^+} \quad (13)$$

$$\frac{\partial \sigma_h^2}{\partial \mu_k} \cong \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\mathbf{x}_{MPP}^-} + \frac{4}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\boldsymbol{\mu}_\mathbf{x}} + \frac{1}{6}\frac{\partial h^2(\mathbf{x})}{\partial x_k}\Big|_{\mathbf{x}=\mathbf{x}_{MPP}^+} - \frac{\partial \mu_h^2}{\partial \mu_k} \quad (14)$$

where $\mathbf{x}_{MPP}^-$ and $\mathbf{x}_{MPP}^+$ is given by transforming the two MPPs obtained from (12) into *X*-space, respectively.

A difference between two moment integration schemes is that quadrature points of the univariate DRM lie on the *x*-axis, whereas quadrature points of PMI lie on the MPP locus as in Fig. 1. Therefore, the number of quadrature points of the univariate DRM increases as the number of design random variables is getting large, whereas the number of quadrature points of PMI does not change since the integration is carried out in the output performance space.

3. Case Studies

To investigate the accuracy and efficiency of the proposed PMI method, two RDO problems are considered. Therein, the design target is to minimize a quality loss function expressed in terms of the first two statistical moments. The first is a ten-dimensional mathematical model, and the second is loudspeaker design problem with seventeen design random variables as a practical EM design problem.

3.1. Mathematical Model

The RDO formulation of a mathematical test problem with ten design random variables is given as follows.

$$\begin{aligned} &\text{minimize } f(\mu_h, \sigma_h^2) = w_1(\mu_h/\mu_{h0})^2 + w_2(\sigma_h/\sigma_{h0})^2 \\ &h(\mathbf{x}) = x_1^{\,2} + x_2^{\,2} + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 \\ &\quad + (x_5-3)^2 + 2(x_6-1)^2 + 5x_7^{\,2} + 7(x_8-11)^2 + 2(x_9-10)^2 \\ &\quad + (x_{10}-7)^2 + 45 \\ &\text{subject to } \ g_1(\mathbf{x}) = 4x_1 + 5x_2 - 3x_7 + 9x_8 - 105 \ge 0 \\ &\quad g_2(\mathbf{x}) = 3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120 \le 0 \\ &\quad g_3(\mathbf{x}) = x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \ge 0 \\ &\quad g_4(\mathbf{x}) = -3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10} \le 0 \end{aligned} \quad (15)$$

where f is the quality loss function defined by the mean μ_h and standard deviation (SD) σ_h of h, and w_1 and w_2 are weight factors of 0.1 and 0.9, respectively. The symbols, μ_{h0} and σ_{h0}, are nominal mean and SD values of h at an initial point. It is assumed that the random variables comply with standard normal probability distributions, and their SD values are shown in Table 2.

Starting with the same initial point, the RDO problem of (15) was solved twice by each of the univariate DRM and the proposed PMI. Performance indicators between three different points are presented in Table 2, where exact mean and SD values of h were obtained from MCS. It is observed that two RDO optima converge towards one point, and their quality loss function values are reduced by nearly 10 % compared to the initial one. The estimated mean and SD values at the two RDO optima show a good agreement to exact ones within the maximum error of 2.74 %. Meanwhile, it is obvious that the proposed PMI saves the computational cost required for the univariate DRM by 53.2 % even though the same iterative designs are carried out during the both RDO processes.

Table 2. Performance Indicators between Two Different Moment Integration Schemes.

		SD	Initial	RDO Univariate DRM	RDO Proposed PMI
x_1		0.1	2.03	1.98	1.98
x_2		0.1	3.15	3.06	3.06
x_3		0.6	8.15	8.59	8.58
x_4		0.1	5.22	5.37	5.34
x_5		0.1	1.04	1.03	1.03
x_6		0.1	1.42	1.41	1.41
x_7		0.6	1.00	1.00	1.00
x_8		0.1	10.95	10.85	10.87
x_9		0.6	7.87	8.65	8.69
x_{10}		0.1	8.70	10.86	10.82
Performance function h			11.41	17.96	16.32
Mean (μ_h)	Estimated	-	-	21.01	20.61
	Exact	-	14.46	21.01	20.55
SD (σ_h)	Estimated	-	-	7.45	7.68
	Exact	-	8.73	7.66	7.65
Quality loss function f		-	1	0.90	0.89
Iterative designs/ Function calls		-	-	4/252	4/118

Exact solutions were recalculated at three different design points by MCS with 1,000,000 samples.

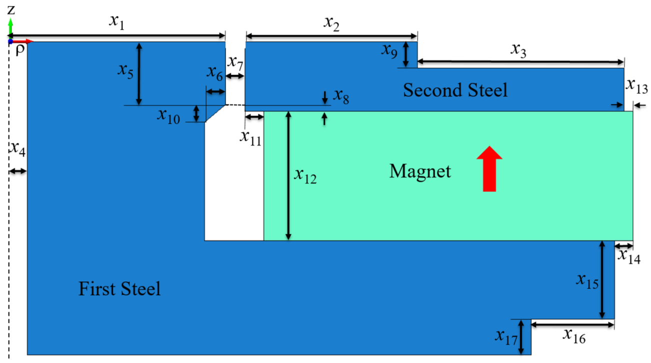

Fig. 2. (Color online) Two-dimensional axisymmetric configuration of a loudspeaker.

Table 3. Properties of Design Random Variables in (16).

Variable	Initial value ($\mu(x_i)$ mm)	SD	Variable	Initial value ($\mu(x_i)$ mm)	SD
x_1	18.95	0.1	x_{10}	1.52	0.05
x_2	17.80	0.1	x_{11}	1.26	0.05
x_3	18.21	0.1	x_{12}	16.22	0.05
x_4	1.44	0.05	x_{13}	0.81	0.01
x_5	5.54	0.05	x_{14}	1.63	0.05
x_6	1.02	0.05	x_{15}	6.86	0.05
x_7	1.73	0.05	x_{16}	7.37	0.05
x_8	0.58	0.01	x_{17}	3.18	0.05
x_9	2.34	0.05	-	-	-

3.2. Loudspeaker Model

Fig. 2 depicts the configuration of a loudspeaker consisting of a steel yoke and a permanent magnet [9]. The design goal is to minimize the quality loss function f concerned with the average air-gap flux density $\mathbf{B}_{ag}$ subject to the constraint for a loudspeaker mass M as in (16).

$$\begin{aligned} &\text{minimize } f(\mu_h, \sigma_h^2) = w_1(\mu_h/\mu_{h0})^2 + w_2(\sigma_h/\sigma_{h0})^2 \\ &\qquad h(\mathbf{x}) = |\mathbf{B}_{ag}(\mathbf{x})| \qquad (16) \\ &\text{subject to } g(\mathbf{x}) = 1 - (M(\mathbf{x}) - M_t)/(0.05 \times M_t) \end{aligned}$$

where weight factors of w_1 and w_2 are set to be 0.1 and 0.9, respectively, and the target mass M_t is 7.5 kg. It is assumed that the seventeen design random variables follow to normal probability distributions, of which SD values are given in Table 3. To take into account a nonlinear B-H curve of the steel yoke, the performance function h was computed by means of a commercial finite element analysis (FEA) tool, called MagNet [10]. The main optimization program was implemented by means of Matlab functions, of which a function call remotely executed MagNet. Therein, the statistical moments and their sensitivities of h were calculated from FEA results as described in [7].

To relieve a heavy computational burden on the RDO process, the deterministic optimum in [9] is considered as an initial design. Two moment values, μ_{h0} and σ_{h0}, in (16) were calculated at the initial design point. Launching at the same point, two RDO optima were obtained by the univariate DRM and proposed PMI, respectively. The two optima are very close to each other, and Fig. 3 compares loudspeaker contours between the initial and RDO loudspeaker designs.

Table 4 shows the performance indicators between three different designs (initial and two RDO designs). The estimated mean and SD values at the two RDO optima show a good agreement to each other, and their quality loss functions are reduced by nearly 70 % compared to the initial one. On the other hand, the same three iterative designs were needed to obtain two RDO designs, whereas

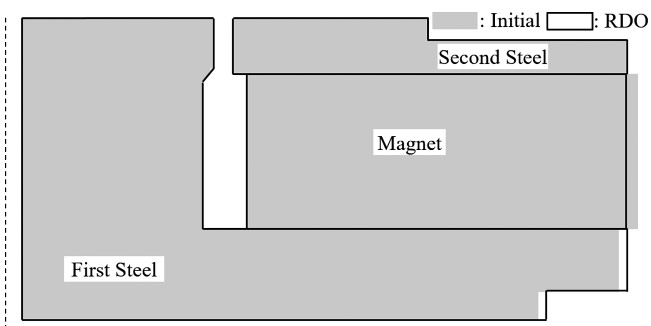

Fig. 3. Contours of two loudspeaker designs.

Table 4. Performance Indicators between Two Different Moment Integration Schemes.

	Initial	RDO	
		Univariate DRM	Proposed PMI
Performance function h	1.803	1.790	1.790
Mean (μ_h)	1.811	1.804	1.791
SD ($_h$)	0.022	0.010	0.011
Quality loss function f	1	0.298	0.323
Iterative designs/FEA calls	-	3/210	3/94

the numbers of FEA simulations make a big difference with each other by 55.2 %. This implies that the proposed PMI is more efficient than the univariate DRM in computing the statistical moments and their sensitivities during the RDO process. However, such numerical improvement in seventeen-dimensional problem does not come up to our expectation when compare to the ten-dimensional mathematical problem. From the fact, it can be deduced that the number of FEA simulations to explore two MPP points in the proposed PMI process increases as the number of design random variables increases.

4. Conclusion

This paper proposes a PMI method to efficiently estimate statistical moments and their sensitivities required for RDO. Results show that the proposed method can substantially save a computational cost without scarifying numerical accuracy, especially when dealing with multi-dimensional EM performance functions.

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