What is electron?

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GIFT, POSTECH

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Contents

History

1600 *De Magnete* (William Gilbert): new *L. electricus*,
L. $<$ *electrum*, *Gk.* $< \bar{\eta}\lambda\varepsilon\kappa\tau\rho o\nu$, **amber**.

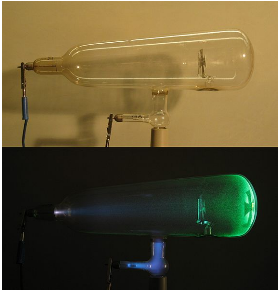
1838 Richard Laming:
$\text{Atom} = \text{core matter} + \sum_{\text{surrounding}} (\text{unit electric charge})$.

1846 William Weber: $\text{Electricity} = \sum \text{fluid}^{(+)} + \sum \text{fluid}^{(-)}$.

1881 Hermann von Helmholtz: “behaves like atoms of electricity.”

1891 George Johnstone Stoney: *electron* $=$ *electr(ic)* $+$ *(i)on*.

Discovery

Crookes tube: 

1869 Johann Wilhelm Hittorf: A glow emitted from the cathode.

1876 Eugen Goldstein: Cathode rays.

1870 Sir William Crookes: The luminescence rays comes from the cathod rays which
- carried energy,
- moved from cathod to anode, and
- bent in magnetic field as negative charged.

1890 Arthur Schuster: The **charge-to-mass ratio,** e/m

Discovery

1892 Hendrik Antoon Lorentz: $\text{mass} \Leftarrow \text{electric charge}$.

1896 J. J. Thomson with John S. Townsend and H. A. Wilson: e/m was independent of cathode material.

1896 George F. Fitzgerald: The universality of e/m and again proposed the name ***electron***.

1896 Henri Becquerel: Radioactivity.

1896 Ernest Rutherford designated the radioactive particles, alpha (α) and beta (β).

1900 Becquerel: The β-rays have the same e/m as electrons.

1909 Robert Millikan and Harvey Fletcher: The oil-drop experiments (published in 1911).

1913 Abram Ioffe confirmed the Millikan's experiments.

Fundamental properties

Mass: $m = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV$/c^2$,
where $c = 2.998 \times 10^8$ m/s.

Charge: $e = -1.602 \times 10^{-19}$ C

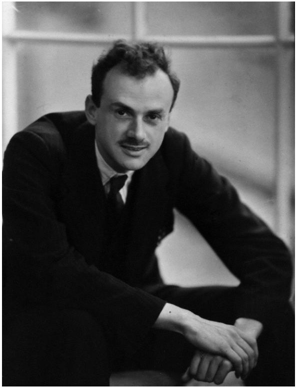
Spin: Intrinsic spin angular momentum with

- $S^2 = s(s+1)\hbar^2$, the square of the spin magnitude, where $s = \pm\frac{1}{2}$ and $\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34}$ Js.
- $\mu = -g\mu_B s$, the spin magnetic moment, where $\mu_B = \frac{e\hbar}{2mc} = 0.927 \times 10^{-20}$ emu and g is the *Landé g-factor*, for free-electron $g = 2.0023$.

Size: A point particle, no larger than 10^{-22} m,

- $r_e = \frac{\alpha\hbar}{mc} = 2.818 \times 10^{-15}$ m, the classical electron radius, where $\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137.04} = 0.00730$, the fine structure constant.
- $\lambda_C = \frac{\hbar}{mc} = 3.862 \times 10^{-13}$ m, the electron Compton wavelength.

Free relativistic quantum fields



A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data. – P. A. M. Dirac

The Schrödinger equation

The time development of a physical system is expressed by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

where the Hamiltonian H is a linear Hermitian operator. For an isolated free particle, the Hamiltonian is

$$H = \frac{p^2}{2m}$$

and the quantum mechanical transcriptions are

$$H \to i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \to \frac{\hbar}{i}\nabla$$

leads to a *relativistically incorrect* equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$

Systems of units

It is convenient to introduce the *natural unit* system for describing relativistic theories.

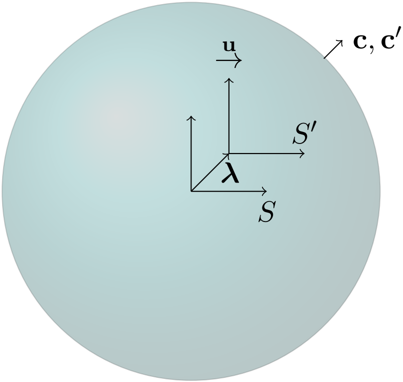
- The natural unit system is defined by the constants $c = \hbar = 1$.
  - In this system,

$$[\mathrm{length}] = [\mathrm{time}] = [\mathrm{energy}]^{-1} = [\mathrm{mass}]^{-1}\,.$$

  - The mass of a particle is equal to the rest energy (mc^2) and to its inverse Compton wavelength $(mc/\hbar)$.
- The thermal unit system is the same as the natural unit system with the additional Boltzmann constant $k_\mathrm{B} = 1$.
  - In this system, $[\mathrm{energy}] = [\mathrm{temperature}]$.
  - Especially, $1\ \mathrm{eV} = 11605\ \mathrm{K}$.
- The atomic Hartree unit system is defined by the constants $\hbar = e^2 = m = 1$, but $c = \alpha^{-1}$.
- The atomic Rydberg unit system is the same as the atomic Hartree unit system, but $2e^2 = 1$.

Special theory of relativity

Einstein concluded that the Maxwell's equations are correct.



So *every* physical law has to satisfy the condition $c = c'$.
The corresponding space-time transformation group is called

- Homogeneous Lorentz group if $\lambda = 0$,
- Poincaré group or inhomogeneous Lorentz group if $\lambda \neq 0$.

Relativistic notions

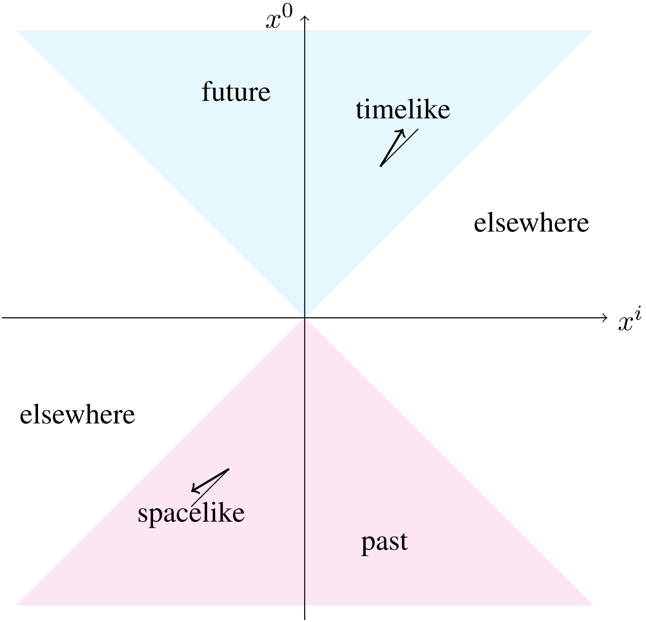
- x is the four-vector of space and time.
- x^μ $(\mu = 0, 1, 2, 3)$ are the *contravariant* components of this vector.
- x_μ are the *covariant* components effected by the Minkowski metric tensor,

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- $x^\mu = \left(x^0, \mathbf{x}\right)$ and $x_\mu = g_{\mu\nu} x^\nu \left(= \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu\right) = \left(x^0, -\mathbf{x}\right)$.
- The scalar product is defined by $x \cdot x \equiv x^\mu x_\mu = t^2 - \mathbf{x}^2$.
- The equation for the *lightcone*: $x^2 \equiv x^\mu x_\mu = 0$.
- The displacement vector is naturally raised, x^μ, while the derivative operator is naturally lowered

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right).$$

Lightcone



Relativistic quantum mechanics

- Momentum vectors are similarly defined $p^\mu = (E, p_x, p_y, p_z)$
- and the scalar product is defined by

$$p \cdot p = p^\mu p_\mu = E^2 - \mathbf{p} \cdot \mathbf{p} = m^2.$$

- Likewise $p \cdot x = p^\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x}$.
- The quantum mechanical transcriptions will be ($\hbar = 1$)

$$E = i\frac{\partial}{\partial x^0}, \quad \mathbf{p} = -i\boldsymbol{\nabla}$$
$$\text{or} \qquad p^\mu = i\partial^\mu.$$

- For a relativistic free particle, we may try a relativistic Hamiltonian $H = \sqrt{p^2 + m^2}$ ($c = 1$) to obtain

$$i\frac{\partial \psi}{\partial t} = \sqrt{-\nabla^2 + m^2}\psi \quad (?)$$

The causality violation

The amplitude for a free particle to propagate from $\mathbf{x}_0$ to $\mathbf{x}$:

$$\begin{aligned} U(t) &= \langle \mathbf{x} | \, e^{-iHt} \, | \mathbf{x}_0 \rangle \\ &= \langle \mathbf{x} | \, e^{-it\sqrt{\mathbf{p}^2+m^2}} \, | \mathbf{x}_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3p \; p e^{it\sqrt{\mathbf{p}^2+m^2}} \; e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\ &= \frac{1}{2\pi^2 \, |\mathbf{x}-\mathbf{x}_0|} \int_0^\infty dp \, p \sin\left(p \, |\mathbf{x}-\mathbf{x}_0|\right) e^{-it\sqrt{p^2+m^2}}. \end{aligned}$$

At the point $x^2 \gg t^2$ (well outside the lightcone), the phase function $px - t\sqrt{p^2+m^2}$ has a stationary point at $p = i\frac{mx}{\sqrt{x^2-t^2}}$. We will have the propagation amplitude as

$$U(t) \sim e^{-m\sqrt{x^2-t^2}},$$

which is small but nonzero *outside* the lightcone.
Causality is violated!

The action principle

The action S in *local field theory* is defined by the time integral of the Lagrangian density $\mathcal{L}$ of the set of the components of the field $\phi_r(x)$ and their derivatives $\partial_\mu \phi_r(x)$:

$$S = \int_{\sigma_0}^{\sigma} \mathcal{L}\left(\phi_r, \partial_\mu \phi_r\right) d^4x,$$

where a general spacelike plane σ at an instance τ is characterized by an equation of plane

$$\sigma : \quad n \cdot x + \tau = 0, \quad n^2 = +1,$$

where n^μ is a unit timelike normal vector.
The variation of the action is

$$\begin{aligned} \delta S &= \int_{\sigma_0}^{\sigma} \left(\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_r\right)} \right) \delta_0 \phi_r d^4x + F(\sigma) - F(\sigma_0), \\ F(\sigma) &= \int_{\sigma} \left[\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_r\right)} \partial^\nu \phi_r - g^{\mu\nu} \mathcal{L} \right) \delta x_\nu - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_r\right)} \delta \phi_r \right] d\sigma_\mu . \end{aligned}$$

The equation of motion

If we choose a variation which vanishes at the boundary planes σ_0 and σ, the observables at the boundary are unchanged for the total variation $\delta\phi_r = \delta_0\phi_r + \partial_\mu\phi_r\delta x^\mu$,

$$\begin{aligned} F(\sigma) &= F(\sigma_0) = 0 \\ \delta S &= \int_{\sigma_0}^{\sigma} \left(\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_r \right)} \right) \delta_0 \phi_r d^4x = 0. \end{aligned}$$

This equation is satisfied if the integrand vanishes at every point:

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_r \right)} = 0.$$

These are the field equations. Note that we may write for the general variation of S simply

$$\delta S = F(\sigma) - F(\sigma_0).$$

Lorentz transformations

The components of a four-vector referred to two different inertial systems with the *same origin* are related by a homogeneous proper Lorentz transformation, which is defined as the real linear transformation which leaves $x^2 = x'^2 = 0$ invariant,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}; \quad \Lambda^{\mu}{}_{\nu} \Lambda^{\lambda\nu} = g^{\mu\lambda},$$

and which, in addition, satisfies

$$\Lambda^{\mu}{}_{\nu} \text{ real}, \quad \det\left(\Lambda^{\mu}{}_{\nu}\right) > 0, \quad \Lambda^{0}{}_{0} > 0.$$

The inhomogeneous Lorentz transformation involves *displacements*, such that $x' = Lx$:

$$L: \quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \lambda^{\mu},$$

where λ^{μ} is a four-vector independent of x. The field component $\phi_r(x)$ transforms according to the proper Lorentz transformation:

$$\phi_r{}'(Lx) = S_r{}^{s} \phi_s(x).$$

Lorentz group

For the Lorentz transformation of the field

$$\phi_r{}'(x) = U(L)\phi_r(x)U^{-1}(L),$$

we observe that the operators $U(L)$ form a representation of the Lorentz group:

$$U(L_2L_1) = U(L_2)U(L_1).$$

The infinitesimal Lorentz transformations are defined by

$$\Lambda_\mu{}^\nu = g_\mu{}^\nu + \alpha_\mu{}^\nu, \quad \lambda_\mu = \alpha_\mu,$$

where $\alpha_\mu{}^\nu$ and α_μ are infinitesimals *of* first order. The relation

$$\Lambda^\mu{}_\nu \Lambda^{\lambda\nu} = g^{\mu\lambda}$$

then leads to

$$\alpha_{\mu\nu} + \alpha_{\nu\mu} = 0.$$

Poincaré group

The infinitesimal part of the transformation U may be written explicitly

$$U = \mathbf{1} + iK,$$

where the *generator* K is written as a linear function of the α's:

$$K = \frac{1}{2} M^{\mu\nu} \alpha_{\mu\nu} + P^{\mu} \alpha_{\mu}, \quad M^{\mu\nu} = -M^{\nu\mu}.$$

Lie's theorem asserts that such operators X_r satisfy

$$[X_r, X_s] = \sum_t c_{rs}{}^t X_t,$$

where the coefficients $c_{rs}{}^t$ are called the structure constants of the group. This relation takes the form

$$\begin{aligned}
[P^{\mu}, P^{\nu}] &= 0, \\
-i \left[M^{\mu\nu}, P^{\lambda} \right] &= g^{\mu\nu} P^{\nu} - g^{\nu\lambda} P^{\mu}, \\
-i \left[M^{\mu\nu}, M^{\rho\sigma} \right] &= g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} + g^{\nu\sigma} M^{\mu\rho} - g^{\nu\rho} M^{\mu\sigma}.
\end{aligned}$$

Poincaré group

The transformed field components ${\phi_r}'$ under the opertator U may be written

$$ {\phi_r}' = U\phi_r U^{-1} = (\mathbf{1} + iK)\,\phi_r\,(\mathbf{1} - iK) \simeq \phi_r + i\,[K, \phi_r]\,. $$

So we have the increment of the field compoents after the transformation

$$ \delta\phi_r = i\,[K, \phi_r]\,. $$

The infinitesimal part of the transformation matrix ${S_r}^s$:

$$ {S_r}^s = {\delta_r}^s + {\Sigma_r}^s, \quad {\Sigma_r}^s = \frac{1}{2}{\Sigma_r}^{s\mu\nu} a_{\mu\nu}, $$

where the coefficients ${\Sigma_r}^{s\mu\nu} = -{\Sigma_r}^{s\nu\mu}$. The increment of ϕ_r becomes

$$ \delta\phi_r(x) = \frac{1}{2}\left[{\Sigma_r}^{s\mu\nu}\phi_s(x) + \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi_r(x)\right]\alpha_{\mu\nu} - \partial^\mu\phi_r(x)\alpha_\mu. $$

Momentum operators

We obtain the defining relations for the momentum operators:

$$
\begin{aligned}
i\left[M^{\mu\nu}, \phi_r(x)\right] &= \Sigma_r{}^{s\mu\nu}\phi_s(x) + \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi_r(x), \\
i\left[P^\mu, \phi_r(x)\right] &= -\partial^\mu\phi_r(x).
\end{aligned}
$$

Under an infinitesimal Lorentz transfromation the plane σ suffers a displacement:

$$\delta x^\mu = \alpha^\mu{}_\nu x^\nu + \alpha^\nu,$$

and the field at the displaced point $x + \delta x$ is $\phi_r + \delta\phi_r$, with

$$\delta\phi_r(x) = \frac{1}{2}\Sigma_r{}^{s\mu\nu}\phi_s(x)\alpha_{\mu\nu}.$$

The generating operator $F(\sigma)$ yields

$$F(\sigma) = \int_\sigma \left[T^{\mu\nu}\left(\alpha_{\nu\rho}x^\rho + \alpha_\nu\right) - \frac{1}{2}\pi^{r\mu}\Sigma_r{}^{s\nu\rho}\phi_s\alpha_{\nu\rho}\right] d\sigma_\mu,$$

where

$$\pi^{r\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_r\right)}, \quad T^{\mu\nu} = \pi^{r\mu}\partial^\nu\phi_r - g^{\mu\nu}\mathcal{L}.$$

Momentum operators

We write $F(\sigma)$ in the form

$$\begin{aligned} F(\sigma) &= \frac{1}{2} M^{\mu\nu} \alpha_{\mu\nu} + P^{\mu} \alpha_{\mu}, \\ M^{\mu\nu} &= \int_{\sigma} \left(T^{\rho\mu} x^{\nu} - T^{\rho\nu} x^{\mu} - \pi^{r\,\rho} \Sigma_{r}{}^{s\mu\nu} \phi_{s} \right) d\sigma_{\rho} \\ P^{\mu} &= \int_{\sigma} T^{\rho\mu} d\sigma_{\rho}. \end{aligned}$$

¶The operator $F(\sigma)$ is the generating operator for the *variation* of the field at a point on the boundary σ: This variation is

$$\begin{aligned} \delta_0 \phi_r &= \frac{1}{2} \left[\Sigma_{r}{}^{s\mu\nu} \phi_s + \left(x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu} \right) \phi_r \right] \alpha_{\mu\nu} - \partial^{\mu} \phi_r \alpha_{\mu} \\ &= i \left[F(\sigma), \phi_r \right], \end{aligned}$$

which must hold for arbitrary values of the ten parameters $\alpha_{\mu\nu}$ and α_{μ}.

Momentum operators

So we obtain the set of equations

$$
\begin{aligned}
i\left[M^{\mu\nu}, \phi_r\right] &= \Sigma_r{}^{s\mu\nu}\phi_s + \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi_r, \\
i\left[P^\mu, \phi_r\right] &= -\partial^\mu\phi_r .
\end{aligned}
$$

The tensor $T^{\mu\nu}$ is called the *canonical momentum tensor*, while the angular momentum tensor $M^{\mu\nu}$ may be split into two parts, defined by

$$
\begin{aligned}
M^{\mu\nu} &= L^{\mu\nu} + N^{\mu\nu} \quad \text{(total angular momentum)}, \\
L^{\mu\nu} &= \int_\sigma \left(T^{\rho\mu}x^\nu - T^{\rho\nu}x^\mu\right) d\sigma_\rho \quad \text{(orbital angular momentum)}, \\
N^{\mu\nu} &= -\int_\sigma \pi^{r\,\rho}\Sigma_r{}^{s\mu\nu}\phi_s d\sigma_\rho \quad \text{(spin angular momentum)}.
\end{aligned}
$$

Conservation laws

For any set of functions $f^\mu(x)$ which vanish sufficiently fast in spacelike directions

$$\int_{\sigma_0}^{\sigma} \partial_\mu f^\mu d^4x = -\int_\sigma f^\mu d\sigma_\mu + \int_{\sigma_0} f^\mu d\sigma_\mu = 0,$$

if the conservation law holds, so that we have

$$\partial_\mu f^\mu = 0.$$

Applying this results to the integrands of $M^{\mu\nu}$ and P^μ, we obtain

$$\begin{aligned} T^{\mu\nu} - T^{\nu\mu} + \partial_\rho H^{\rho\mu\nu} &= 0, \\ \partial_\mu T^{\mu\nu} &= 0, \\ \text{with} \quad H^{\rho\mu\nu} &= \pi^{r\,\rho} \Sigma_r{}^{s\mu\nu} \phi_s = -H^{\rho\nu\mu}. \end{aligned}$$

Conservation laws

Defining the symmetrical momentum tensor

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\rho\mu\nu},$$

where

$$G_{\rho\mu\nu} = \frac{1}{2}\left(H_{\rho\mu\nu} + H_{\mu\nu\rho} + H_{\nu\mu\rho}\right),$$

we obtain the following tensor properties

$$\begin{aligned}
\Theta^{\mu\nu} &= \Theta^{\nu\mu}, \\
P^{\nu} &= \int_\sigma \Theta^{\mu\nu} d\sigma_\mu, \\
\partial^\mu \Theta_{\mu\nu} &= 0.
\end{aligned}$$

Commutation rules

The generating operator is given by

$$F(\sigma) = -\int_\sigma \pi^{r\mu} \delta\phi_r d\sigma_\mu,$$

and the arbitrary variation of the field components are

$$\delta\phi_r(x) = i\left[\phi_r(x), \int_\sigma \pi^{s\mu}(y)\delta\phi_s(y)d\sigma_\mu\right], \quad \text{for } x \in \sigma.$$

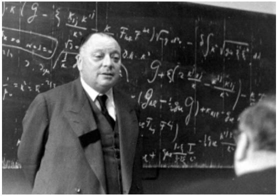
For any three operators A, B, and C, the Jacobi identity is

$$\begin{aligned} [A, BC] &= [A, B] C + B [A, C] \\ &= \{A, B\} C - B \{A, C\}. \end{aligned}$$

So that for the three operators $\phi_r(x)$, $\pi^{s\mu}(y)$, and $\delta\phi_s(y)$, we have two possibilities

$$\begin{aligned} &(a) \qquad [\phi_r(x), \delta\phi_s(y)] = 0, \quad [\phi_r(x), \pi^{s\mu}(y)] = -\delta_r{}^s \delta^\mu(x, y), \\ &(b) \qquad \{\phi_r(x), \delta\phi_s(y)\} = 0, \quad \{\phi_r(x), \pi^{s\mu}(y)\} = -\delta_r{}^s \delta^\mu(x, y). \end{aligned}$$

Pauli's principle



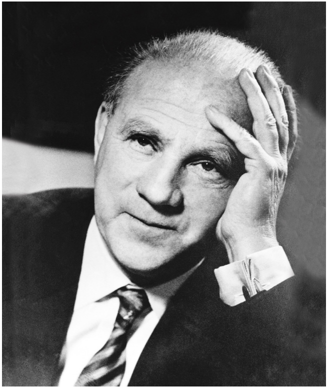
Wolfgang Pauli (1936, 1940) suggested the new principles that

1. The total energy of the system must be a positive definite operator such that the vacuum state is the state of the lowest energy.
2. Observerbles at two points with space-like separation must commute with each other.

The quantization of the fields

- with half-integer spin according to case (a) would violate principle 1, $\rightarrow$ known as the **exclusion principle**,
- while with integer spin according to case (b) would violate principle 2.

Free field quantizations



We have to remember that what we oberve is not nature herself, but nature exposed to our method of questioning. – Werner Heisenberg.

The Klein-Gordon attempts

Considering the Lagrangian of a scalar field

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2,\end{aligned}$$

we obtain the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0,\ \left(\partial^\mu\partial_\mu + m^2\right)\phi = 0,\ \text{or}\ \left(\Box + m^2\right)\phi = 0.$$

Noting that the conjugate to $\phi(x)$ is $\pi(x) = \dot{\phi}(x)$, we can construct the Hamiltonian:

$$H = \int d^3x\mathcal{H} = \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2\right].$$

The Klein-Gordon attempts

In the momentum space representation, the Klein-Gordon field is expanded as

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

so that the Klein-Gordon equation will be

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right] \phi(\mathbf{p}, t) = 0 \quad \text{or} \quad \left[\frac{\partial^2}{\partial t^2} + {\omega_{\mathbf{p}}}^2\right] \phi(\mathbf{p}, t) = 0,$$

which is a simple harmonic oscillator (SHO) equation which can be easily solved by introducing the annihilation and creation operators such that

$$\left[a_{\mathbf{p}}, {a_{\mathbf{p}'}}^\dagger\right] = (2\pi)^3 \delta^{(3)}\left(\mathbf{p} - \mathbf{p}'\right).$$

The Klein-Gordon attempts

We will expand the field $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ in terms of the annihilation and creation operators as

$$\begin{aligned}
\phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}{}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}; \\
\pi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} (i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}{}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}.
\end{aligned}$$

These expansions yield the field commutator relation

$$\begin{aligned}
\left[\phi(\mathbf{x}), \pi(\mathbf{x}')\right] &= \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{-i}{2} \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \\
&\quad \times \left(\left[a_{-\mathbf{p}}{}^{\dagger}, a_{\mathbf{p}'} \right] - \left[a_{\mathbf{p}}, a_{\mathbf{p}'}{}^{\dagger} \right] \right) e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \\
&= i\delta^{(3)}\left(\mathbf{x} - \mathbf{x}'\right).
\end{aligned}$$

The Klein-Gordon attempts

Then the Hamiltonian will be

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{p}} + \underbrace{\frac{1}{2}\left[a_{\mathbf{p}}, a_{\mathbf{p}}{}^{\dagger}\right]}_{=0}\right).$$

The total momentum operator is written as

$$\mathbf{P} = -\int d^3x\pi\left(\mathbf{x}\right)\nabla\phi\left(\mathbf{x}\right) = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{p}}.$$

- ⇒ The operator $a_{\mathbf{p}}{}^{\dagger}$ creates momenum $\mathbf{p}$ and energy $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}| + m^2}$.
- ⇒ The state $a_{\mathbf{p}}{}^{\dagger}a_{\mathbf{q}}{}^{\dagger}\cdots|0\rangle$ has momentum $\mathbf{p}+\mathbf{q}+\cdots$.
- ⇒ We call these excitations *particles*.
- ⇒ We will refer to $\omega_{\mathbf{p}}$ as $E_{\mathbf{p}} = +\sqrt{|\mathbf{p}|^2 + m^2}$, since it is the positive energy of the particle.

The Klein-Gordon attempts

The one-particle state $|\mathbf{p}\rangle \propto a_{\mathbf{p}}{}^{\dagger} |0\rangle$ is normalized with the Lorentz invariance with a boost $p'{}_i = \gamma (p_i + \beta E)$ and $E' = \gamma (E + \beta p_i)$, where $\beta = v_i/c$ and $\gamma = \sqrt{1-\beta^2}$:

$$\begin{aligned}
\delta^{(3)}(\mathbf{p}-\mathbf{q}) &= \delta^{(3)}(\mathbf{p}'-\mathbf{q}') \frac{dp'{}_i}{dp_i} \\
&= \delta^{(3)}(\mathbf{p}'-\mathbf{q}') \gamma \left(1 + \beta \frac{dE}{dp_i}\right) \\
&= \delta^{(3)}(\mathbf{p}'-\mathbf{q}') \frac{\gamma}{E} (E + \beta p_i) \\
&= \delta^{(3)}(\mathbf{p}'-\mathbf{q}') \frac{E'}{E}.
\end{aligned}$$

We define

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}{}^{\dagger} |0\rangle \to \langle \mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}).$$

Time-evolution of the Klein-Gordon fields

The Heisenberg picture of the fields

$$\begin{aligned}\phi(x) &= \phi(\mathbf{x}, t) = e^{iHt}\phi(\mathbf{x})\, e^{-iHt} \\ \pi(x) &= \pi(\mathbf{x}, t) = e^{iHt}\pi(\mathbf{x})\, e^{-iHt}\end{aligned}$$

exhibit the time-evolution by the Heisenberg equation of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$. The time dependences of the annihilation and creation operators are

$$a_{\mathbf{p}}^{\mathrm{H}} \equiv e^{iHt}a_{\mathbf{p}}e^{-iHt} = a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}, \quad a_{\mathbf{p}}^{\mathrm{H}\dagger} \equiv e^{iHt}a_{\mathbf{p}}{}^{\dagger}e^{-iHt} = a_{\mathbf{p}}{}^{\dagger}e^{iE_{\mathbf{p}}t}.$$

Omitting the superscript H, $a_{\mathbf{p}}^{\mathrm{H}} \to a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\mathrm{H}\dagger} \to a_{\mathbf{p}}{}^{\dagger}$, the fields are expanded by the operators:

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}{}^{\dagger}e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}}; \\ \pi(\mathbf{x}, t) &= \frac{\partial}{\partial t}\phi(\mathbf{x}, t):\end{aligned}$$

the explicit description of the particle-wave duality.

The Dirac field

- The Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

  has resolved the relativistic inconsistency of the Schrödinger equation.
- However, the quantization $\left[a, a^\dagger\right] = 1 \nRightarrow$ electron.
- Dirac (1928) suggested another Lagrangian:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi. \quad (\bar{\psi} \equiv \psi^\dagger\gamma^0)$$

- The canonical momentum conjugate to ψ is $i\psi^\dagger$,
- and thus the Hamiltonian is

$$H = \int d^3x\bar{\psi}\left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right)\psi = \int d^3x\psi^\dagger\left[-i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\gamma^0\right]\psi$$

$$= \int d^3x\bar{\psi}\left[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta\right]\psi. \quad (\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma},\ \beta \equiv \gamma^0)$$

Dirac matrices

The Dirac matrices follows the algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{4\times 4} \to S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

Define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma.$$

There are 5 standard classes of the γ-matrices

1	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1
		16

Explicitly, we have

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Dirac spinor

Pauli spin matrices are defined by the Dirac algebra

$$\gamma^j \equiv i\sigma^j \quad \Rightarrow \quad \{\gamma^i, \gamma^j\} = -2\delta^{ij}.$$

The Lorentz algebra are then

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k,$$

the two-dimensional representation of the rotation group.
The boost and rotation generators are

$$\begin{aligned} S^{0i} &= \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \\ S^{ij} &= \frac{i}{4}\left[\gamma^i, \gamma^j\right] = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k, \end{aligned}$$

which transform the four-component field ψ, a *Dirac spinor*.

Dirac equation

The action principle yields the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0.$$

This implies the Klein-Gordon equation shown by acting $(-i\gamma^\mu\partial_\mu - m)$ on the left

$$\begin{aligned}
(-i\gamma^\mu\partial_\mu - m)\,(i\gamma^\nu\partial_\nu - m)\,\psi &= \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi \\
&= \left(\frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\}\partial_\mu\partial_\nu + m^2\right)\psi \\
&= \left(\partial^2 + m^2\right)\psi = 0.
\end{aligned}$$

Since the canonical momentum conjugate to ψ is $i\psi^\dagger$, the Hermitian conjugate form of the Dirac equation is

$$-i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0.$$

Solutions of the Dirac equation

Since a Dirac field ψ obeys the Klein-Gordon equation, we can expand it as linear combinations of plane waves:

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad \psi(x) = v(p)e^{+ip\cdot x}.$$

Plugging them into the Dirac equation, we obtain

$$\begin{aligned}
(\gamma^\mu p_\mu - m)\,u(p) &= \left(\not{p} - m\right)u(p) = 0, \qquad & p^2 = m^2, \quad p^0 > 0,\\
(\gamma^\mu p_\mu + m)\,v(p) &= \left(\not{p} + m\right)v(p) = 0, \qquad & p^2 = m^2, \quad p^0 > 0.
\end{aligned}$$

In the rest frame, with $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, the column vectors $u(p)$ and $v(p)$ are in the form

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}, \quad s = 1, 2,$$

where ξ^s and η^s are the bases of the two-component spinors.

Spin sums

The solutions are normalized accoring to

$$\bar{u}^r(p)u^s(p) = +2m\delta^{rs}, \qquad u^{s\dagger}(p)u^s(p) = +2E_{\mathbf{p}}\delta^{rs},$$
$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs}, \qquad v^{s\dagger}(p)v^s(p) = +2E_{\mathbf{p}}\delta^{rs}.$$

The u's and v's are orthogonal to each other:

$$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0,$$

but

$$u^{r\dagger}(\mathbf{p})\, v^s(\mathbf{p}) = v^{r\dagger}(-\mathbf{p})\, u^s(\mathbf{p}) = 0.$$

Then the completeness relations are

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \gamma \cdot p + m = \gamma^\mu p_\mu + m = \not{p} + m, \\ \sum_s v^s(p)\bar{v}^s(p) &= \gamma \cdot p - m = \gamma^\mu p_\mu - m = \not{p} - m. \end{aligned}$$

The quantized Dirac field

The Dirac field operators are expanded by plane waves

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\,\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right);$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\,\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} \right),$$

where the creation and annihilation operators obey the anticommutation relations

$$\left\{ a^r_{\mathbf{p}}, a^{s\,\dagger}_{\mathbf{q}} \right\} = \left\{ b^r_{\mathbf{p}}, b^{s\,\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \, \delta^{(3)} \left(\mathbf{p} - \mathbf{q} \right) \delta^{rs}.$$

The equal-time anticommutation relations for ψ and $\psi^\dagger$ are then

$$\left\{ \psi_a \left(\mathbf{x} \right), \psi_b{}^\dagger \left(\mathbf{y} \right) \right\} = \delta^{(3)} \left(\mathbf{x} - \mathbf{y} \right) \delta_{ab};$$

$$\left\{ \psi_a \left(\mathbf{x} \right), \psi_b \left(\mathbf{y} \right) \right\} = \left\{ \psi_a{}^\dagger \left(\mathbf{x} \right), \psi_b{}^\dagger \left(\mathbf{y} \right) \right\} = 0.$$

Physical meaning of the Dirac field

The vacuum $|0\rangle$ is defined to be the state such that

$$a_{\mathbf{p}}^{s}|0\rangle = b_{\mathbf{p}}^{s}|0\rangle = 0.$$

The Hamiltonian, with dropping the infinities, are written

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left(a_{\mathbf{p}}^{s\,\dagger} a_{\mathbf{p}}^{s} + b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^{s} \right).$$

The momentum operator is

$$\mathbf{P} = \int d^3x \psi^\dagger (-i\boldsymbol{\nabla}) \psi = \int \frac{d^3p}{(2\pi^3)} \sum_s \mathbf{p} \left(a_{\mathbf{p}}^{s\,\dagger} a_{\mathbf{p}}^{s} + b_{\mathbf{p}}^{s\,\dagger} b_{\mathbf{p}}^{s} \right).$$

Thus both $a_{\mathbf{p}}^{s\,\dagger}$ and $b_{\mathbf{p}}^{s\,\dagger}$ create particles with energy $+E_{\mathbf{p}}$ and momentum $\mathbf{p}$. The one-particle states $|\mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{s\,\dagger} |0\rangle$ is defined so that $\langle \mathbf{p}, r | \mathbf{q}, s\rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs}$ is Lorentz invariant.

Conservations of the Dirac field

The Dirac field transforms according to

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}} \psi\left(\Lambda^{-1} x\right).$$

The change in the field at a fixed point is ($\Lambda_{\frac{1}{2}} \simeq i - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu} = 1 - \frac{i}{2}\theta\Sigma^3$, i.e. infinitesimal rotation angle θ about z-axis)

$$\begin{aligned} \delta\psi &= \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}} \psi\left(\Lambda^{-1} x\right) - \psi(x) \\ &= \left(1 - \frac{i}{2}\theta\Sigma^3\right) \psi\left(t, x + \theta y, y - \theta x, z\right) - \psi(x) \\ &= -\theta\left(x\partial_y + y\partial_x + \frac{i}{2}\Sigma^3\right)\psi(x) \equiv \theta\Delta\psi. \end{aligned}$$

The conserved Noether currents are

$$\begin{aligned} j^0 &= \frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi\right)}\Delta\psi = -i\bar{\psi}\gamma^0\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi, \\ \mathbf{J} &= \int d^3x\psi^\dagger\left(\mathbf{x}\times(-i\nabla) + \frac{1}{2}\mathbf{\Sigma}\right)\psi. \end{aligned}$$

Spin-$\frac{1}{2}$ Dirac field

At $t=0$, for simplicity,

$$\begin{aligned} J_z &= \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}'}}} e^{-i\mathbf{p}'\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \\ & \quad \times \sum_{r,r'} \left(a^{r'\,\dagger}_{\mathbf{p}'} u^{r'}(\mathbf{p}') + b^{r'}_{-\mathbf{p}'} v^{r'\,\dagger}(-\mathbf{p}') \right) \\ & \qquad \times \frac{\Sigma^3}{2} \left(a^r_{\mathbf{p}} u^r(\mathbf{p}) + b^{r}_{-\mathbf{p}}{}^{\dagger} v^r(-\mathbf{p}) \right) . \end{aligned}$$

The commutator rules for $a^{s\,\dagger}_0$ yields

$$J_z a^{s\,\dagger}_0 |0\rangle = \frac{1}{2m} \sum_r \left(u^{s\,\dagger}(0) \frac{\Sigma^3}{2} u^r(0) \right) a^{r\,\dagger}_0 |0\rangle = \sum_r \left(\xi^{s\,\dagger} \frac{\sigma^3}{2} \xi^r \right) a^r_0 |0\rangle ;$$

the eigenvalues of J_z are $\pm\frac{1}{2}$.
$\Rightarrow$ The Dirac field conveys spin-$\frac{1}{2}$.

Conserved quantities of the Dirac field

- A current $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ is conserved by the Dirac equation

$$\begin{aligned}\partial_\mu j^\mu &= \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \\ &= \left(im\bar{\psi}\right)\psi + \bar{\psi}\left(-im\psi\right) = 0.\end{aligned}$$

- The charge associated with this current is

$$Q = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_{\mathbf{p}}^{s\,\dagger}a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\,\dagger}b_{\mathbf{p}}^s\right)$$

  is conserved: there is a unit charge e.
- An axial vector current $j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ is conserved

$$\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi.$$

  if $m = 0$.

Discrete symmetries of the Dirac field

Let C the charge conjugation, P the parity, T the time reversal operators. Use the shorthand $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu \equiv -1$ for $\mu = 1, 2, 3$.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

- The free Dirac Lagrangian $\mathcal{L}_0 = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ is invariant under C, P, and T separately.
- The perturbation $\delta\mathcal{L}$ must be a Lorentz scalar.
- All Lorentz scalar combinations of $\bar{\psi}$ and ψ are invariant under the combined symmetry CPT.

Propagators and causality

The amplitude for a scalar Klein-Gordon particle to propagate from y to x is $\langle 0|\,\phi(x)\phi(y)\,|0\rangle$:

$$\underset{y}{\cdot}\dashrightarrow\underset{x}{\cdot} \overset{?}{=} \langle 0|\,\phi(x)\phi(y)\,|0\rangle = D'(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}.$$

- When $x-y$ is purely in the time direction, $(x-y)^2 > 0$; $x^0 - y^0 = t$, $\mathbf{x}-\mathbf{y} = 0$:

$$\begin{aligned} D'(x-y) &= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}}e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}e^{-iEt} \\ &\underset{t\to\infty}{\sim} e^{-imt}. \end{aligned}$$

Propagators and causality

- When $x-y$ is purely spatial direction, $(x-y)^2 < 0$; $x^0 - y^0 = 0$, $\mathbf{x} - \mathbf{y} = \mathbf{r}$:

$$
\begin{aligned}
D'(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{r}} \\
&= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\mathbf{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr} \\
&= \frac{-i}{2\,(2\pi)^2\, r} \int_{-\infty}^{\infty} dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}} \\
&\underset{r\to\infty}{\sim} e^{-mr}.
\end{aligned}
$$

$\Rightarrow$ Causality is still violated so we need to a correct form of the amplitude vanishing for $(x-y)^2 < 0$.
$\Rightarrow$ Since $\phi(x)$ is a quantum field, let us consider a commutator $[\phi(x), \phi(y)]$.

Propagators and causality

- The amplitude for the commutator

$$\begin{aligned}\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle &= \langle 0|\phi(x)\phi(y)|0\rangle - \langle 0|\phi(y)\phi(x)|0\rangle \\ &= D'(x-y) - D'(y-x) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)\end{aligned}$$

  preserves causality under the Lorentz transformation by taking $(x-y)\to-(x-y)$ on the second term to cancel each other.
- The amplitude integral can convey the frequency integral through the residue theorem:

$$\begin{aligned}\oint \frac{dp_0}{2\pi}\frac{e^{-ip_0(x^0-y^0)}}{p^2-m^2} &= \wp\int \frac{dp_0}{2\pi}\frac{e^{-ip_0(x^0-y^0)}}{{p_0}^2-{E_{\mathbf{p}}}^2} \\ &= -2\pi i\left(\mathrm{Res}|_{p_0=+E_{\mathbf{p}}} + \mathrm{Res}|_{p_0=-E_{\mathbf{p}}}\right) \\ &= -i\frac{1}{2E_{\mathbf{p}}}\left(e^{-iE_{\mathbf{p}}(x^0-y^0)} - e^{+iE_{\mathbf{p}}(x^0-y^0)}\right).\end{aligned}$$

Propagators and causality

The amplitude for $x^0 > y^0$ is then

$$\langle 0| \left[\phi(x), \phi(y)\right] |0\rangle \underset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}.$$

Let us define a function

$$D_R(x-y) \equiv \theta\left(x^0 - y^0\right) \langle 0| \left[\phi(x), \phi(y)\right] |0\rangle \,,$$

which satisfies the Klein-Gordon Green's function equation

$$\begin{aligned} \left(\partial^2 + m^2\right) D_R(x-y) &= -i\delta^{(4)}(x-y), \\ \text{or } \left(-p^2 + m^2\right) \tilde{D}_R(p) &= -i. \end{aligned}$$

which is known as the *retarded* Green's function, explicitly

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}.$$

For $x^0 < y^0$, we have the *advanced* Green's function

$$D_A(x-y) = \theta\left(y^0 - x^0\right) \langle 0| \left[\phi(x), \phi(y)\right] |0\rangle = -D_R(x-y).$$

Propagators and causality

Let us define the Klein-Gordon *Feynman propagator* as

$$\begin{aligned} D'_F(x-y) &\equiv \langle 0| \hat{T}\left\{\phi(x)\phi(y)\right\}|0\rangle \\ &= D_R(x-y) + D_A(y-x) \\ &\underset{(t\equiv x^0-y^0)}{=} \theta(t)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(-t)\langle 0|\phi(y)\phi(x)|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}. \end{aligned}$$

where $\hat{T}\{\cdots\}$ is the *time-ordering* operator, and the integrand of the last line has the poles $p^0 = \pm(E_{\mathbf{p}} - i\epsilon)$, for $\epsilon \to 0^+$.
Similiarly, we can also define the Dirac Feynman propagator as

$$\begin{aligned} S'_F(x-y) &\equiv \langle 0| \hat{T}\left\{\psi(x)\bar{\psi}(y)\right\}|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i\left(\not{p}+m\right)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}. \end{aligned}$$

Electromagnetic interaction

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature . . . –Richard P. Feynman.

Electromagnetic interaction

- We have understood the spin and dynamics of electron as a free Dirac field.
- However, a free particle is not measurable so we need interaction to really observe it.
- An electron is subjected to the electromagnetic interaction with the Lagrangian such that

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}} \\ &= \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e_0\bar{\psi}\gamma^{\mu}\psi A_{\mu}, \end{aligned}$$

where A_μ is the electromagnetic vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the skew-symmetrical electromagnetic field tensor, and $e_0 < 0$ is the electron charge.
- Introducing $D_\mu \equiv \partial_\mu + ie_0 A_\mu$ we have a simpler form

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(iD\!\!\!\!/ - m_0\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}.$$

Electromagnetic interaction

- The QED Lagrangian is invariant under the *gauge transformations*

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x).$$

- The equation of motion for ψ is

$$\left(i\not{D} - m_0\right)\psi(x) = 0,$$

  which is jus the Dirac equation coupled to the electromagnetic field.
- The equation of motion for A_ν is

$$\partial_\mu F^{\mu\nu} = e_0\bar{\psi}\gamma^\nu\psi = e_0 j^\nu,$$

  which is the inhomogeneous Maxwell equations, with the current density $j^\nu = \bar{\psi}\gamma^\nu\psi$.
- The quantization of A_μ fields are depending on the choice of gauges, such as the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ or the Lorentz gauge $\partial_\mu A^\mu = 0$.

Maxwell field

In the relativistic notations, the maxwell field is defined by

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\nu, \quad j^\mu = (\rho, \mathbf{j}) \\ \Rightarrow \quad & \mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \end{aligned}$$

and written as

$$F^{\mu\nu} = -F^{\nu\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

The four-vector potential A^μ does not determined uniquely for a *gauge transformation*

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x),$$

but it yields the Lorentz invariant Maxwell equation

$$\Box A^\mu - \partial^\mu (\partial \cdot A) = j^\mu.$$

Radiation field

- We modify the Lagrangian $\mathcal{L}_{\text{Maxwell}}$ to

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\xi\left(\partial \cdot A\right)^2$$

so that the Maxwell's equation are replace by

$$\Box A_\mu - (1-\xi)\partial_\mu\left(\partial \cdot A\right) = 0$$

and the conjugate momenta π^μ to A_μ are

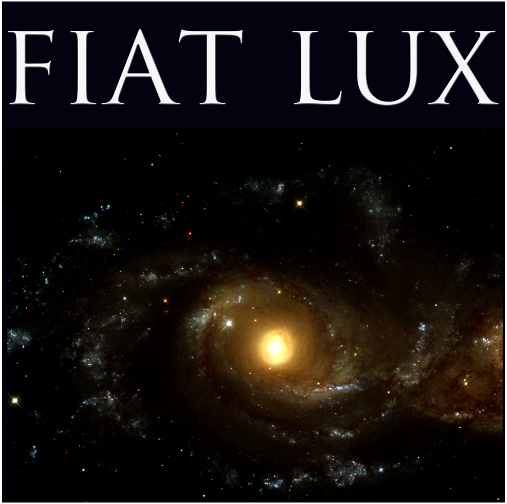
$$\pi^\rho = \frac{\partial \mathcal{L}_{\text{Maxwell}}}{\partial\left(\partial_0 A_\mu\right)} = F^{\mu 0} - \xi g^{\mu 0}\left(\partial \cdot A\right),$$

where $(\partial \cdot A)$ is a scalar field such that $\Box\left(\partial \cdot A\right) = 0$.

- For radiation field, we conveniently choose $\xi = 1$ (Feynman gauge) to yield the Maxwell equation

$$\Box A^\mu = 0.$$

Dixitque Deus,



et facta est lux. – Genesis 1:3

Radiation field

The solutions of $\Box A^\mu = 0$ are the plane waves:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \times \sum_{\lambda=0}^{3} \left[a^{(\lambda)}(k)\varepsilon_\mu^{(\lambda)}(k)e^{-ik\cdot x} + a^{(\lambda)\dagger}(k)\varepsilon_\mu^{(\lambda)*}(k)e^{+ik\cdot x} \right],$$

where $\varepsilon^{(\lambda)}$ are the bases of polarization vectors, which satisfies

$$\sum_\lambda \frac{\varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)}{\varepsilon^{(\lambda)}(k)\cdot\varepsilon^{(\lambda)*}(k)} = g_{\mu\nu}, \quad \varepsilon^{(\lambda)}(k)\cdot\varepsilon^{(\lambda')*}(k) = g^{\lambda\lambda'}.$$

Real photons convey only the *transverse* polarizations $\varepsilon^\mu = (0, \boldsymbol{\varepsilon})$, where $\mathbf{k}\cdot\boldsymbol{\varepsilon} = 0$. For $\mathbf{k} \parallel \hat{\mathbf{z}}$, the right- and left-handed polarization vectors are

$$\varepsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0).$$

Photon: quantized radiation field

The equal-time commutation rules for the radiation field are

$$\begin{aligned} [A_\mu(\mathbf{x}), A_\nu(\mathbf{y})] &= \left[\dot{A}_\mu(\mathbf{x}), \dot{A}_\nu(\mathbf{y})\right] = 0, \\ \left[\dot{A}_\mu(\mathbf{x}), A_\nu(\mathbf{y})\right] &= i g_{\mu\nu} \delta^{(3)}(\mathbf{x}-\mathbf{y}). \end{aligned}$$

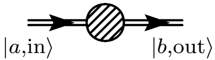
We can define the photon Feynman propagator as

$$\begin{aligned} i g_{\mu\nu} \Delta'_F(x-y) &\equiv \langle 0| \hat{T} \left[A_\mu(x) A_\nu(y)\right] |0\rangle \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\mu\nu}}{k^2 + i\epsilon} e^{-ik\cdot(x-y)} \\ (\text{arbitrary } \xi) &\Rightarrow \int \frac{d^4k}{(2\pi)^4} \left[\frac{-i g_{\mu\nu}}{k^2+i\epsilon} + \frac{1-\xi}{\xi} \frac{-i k_\mu k_\nu}{(k^2+i\epsilon)^2}\right] e^{-ik\cdot(x-y)}. \end{aligned}$$

- Feynman gauge: $\xi = 1$ and Landau gauge: $\xi \to \infty$.
- The longitudinal polarization state could be cured by introduing a fictitious photon mass $\mu \to 0$.

A generic experiment

- A generic experiment is understood diagramatically



- The amplitude $\langle b, \mathrm{out}|a, \mathrm{in}\rangle$ describes the probability that $|a\rangle$ will evolve *in* time and be measured in the $|b\rangle$ state.
- For the incoming state $|i, \mathrm{in}\rangle$, the transition probability to a final state $|f, \mathrm{out}\rangle$ is

$$w_{f\leftarrow i} = |\langle f, \mathrm{out}|i, \mathrm{in}\rangle|^2 \, .$$

- There is a unitary operator, $S^\dagger S = SS^\dagger = 1$, S-matrix:

$$\langle f, \mathrm{out}|i, \mathrm{in}\rangle = \langle f, \mathrm{in}|\, S \,|i, \mathrm{in}\rangle = \langle f, \mathrm{out}|\, S \,|i, \mathrm{out}\rangle \, ,$$

  where $S = 1 + i\tau$ and $S^\dagger = 1 - i\tau$, where the τ-matrix contains the information on the interactions.
- The τ-matrix is consist of the energy-momentum conservation and the *invariant matrix element* $\mathcal{M}$:

$$\langle f|\, i\tau \,|i\rangle = (2\pi)^4 \, \delta^{(4)} \left(P_i - P_f\right) \cdot i\mathcal{M} \left(i \to f\right) .$$

Total decay rate

Consider a reaction of decay

$$a \to 1 + 2 + \cdots + n_f \quad (\text{eg., } \mathrm{Ne}_{2P_4} \to \mathrm{Ne}_{\mathrm{g.s.}} + \gamma).$$

The transition probability per unit time is

$$w_{f\leftarrow i} = \frac{|S_{fi}|^2}{T}.$$

In a cubic box of volume $V = L^3$ with infinitely high potential well, the differential transition probability is

$$dw_{f\leftarrow i} = \frac{1}{(2\pi)^{3n_f-4}} \frac{1}{2E_a} \delta^{(4)}\left(p_f - p_a\right) |\mathcal{M}_{fi}|^2 \prod_f \frac{d^3 p_f}{2E_f}.$$

The lifetime $\tau_a \left(= {\Gamma_a}^{-1}\right)$ is the inverse of the total decay width

$$\begin{aligned} \Gamma_a &= \sum_{n_f} \Gamma_{a\to\{n_f\}} = \sum_{n_f} w_{\{n_f\}\leftarrow a} \\ &= \frac{1}{2E_a} \frac{1}{(2\pi)^{3n_f-4}} \int \frac{d^3 p_1}{2E_1} \cdots \frac{d^3 p_{n_f}}{2E_{n_f}} \delta^{(4)}\left(p_f - p_i\right) |\mathcal{M}_{fi}|^2 . \end{aligned}$$

Differential cross section

Consider a scattering process

$$a + b \to 1 + 2 + \cdots + n_f \quad (\text{eg., } \mathrm{Ne}_{3S_2} + \gamma \to \mathrm{Ne}_{2P_4} + 2\gamma).$$

The transition rate (transition probability per unit time) density to one definite final state is

$$\bar{w}_{f\leftarrow i} = \lim_{V\to\infty} \frac{w_{f\leftarrow i}}{V} = (2\pi)^4 \, \delta^{(4)} \left(P_i - P_f\right) \left|\mathcal{M}_{fi}\right|^2 .$$

The differential cross sectin (in Lab.) is defined as the transition rate density per target density (n_t) per incident flux (F)

$$d\sigma_{fi} = \frac{\bar{w}_{f\leftarrow i}}{n_t F} \prod_{f=1}^{n_f} \frac{d^3 p_f{}'}{(2\pi)^3 \, 2\omega_{p_f{}'}} .$$

The target density $n_t = 2\omega_{p_2}$ and the flux $F = 2\omega_{p_1} v_{\mathrm{rel}}$ yield

$$d\sigma_{fi} = \frac{1}{2\omega_{p_1} 2\omega_{p_2} v_{\mathrm{rel}}} \prod_{f=1}^{n_f} \frac{d^3 p_f{}'}{(2\pi)^3 \, 2\omega_{p_f{}'}} \, (2\pi)^4 \, \delta^{(4)} \left(P_i - P_f\right) \left|\mathcal{M}_{fi}\right|^2 .$$

Interaction picture

- Let $|\Omega\rangle$ be the ground state of the interacting theory.
- Let $H_{\text{int}}(t) = \int d^3x \mathcal{H}_{\text{int}} = -\int d^3x \mathcal{L}_{\text{int}}$ be the interacting Hamiltonian and $H = H_0 + \lambda H_{\text{int}}$ with $0 \leq \lambda \leq 1$.
- Let $\phi(x) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$ be an Heisenberg picture field and for $t \neq t_0$, $\phi(t, \mathbf{x}) = e^{iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH(t-t_0)}$.
- For $\lambda = 0$, H becomes H_0 and we can define an *interaction picture* field as

$$\phi(t, \mathbf{x})\big|_{\lambda=0} = e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} \equiv \phi_I(t, \mathbf{x}) \,.$$

- The full Heisenberg picture field ϕ in terms of ϕ_I:

$$\begin{aligned} \phi(t, \mathbf{x}) &= e^{iH(t-t_0)} \left\{ e^{iH_0(t-t_0)} \phi_I(t, \mathbf{x}) e^{-iH_0(t-t_0)} \right\} e^{-iH(t-t_0)} \\ &\equiv U^\dagger(t, t_0) \, \phi_I(t, \mathbf{x}) \, U(t, t_0) \,, \end{aligned}$$

  where we have defined the unitary operator

$$U(t, t_0) \equiv e^{iH_0(t-t_0)} e^{-iH(t-t_0)} .$$

Unitary time-evolution operator

- The initial condition is $U(t_0, t_0) = 1$.
- The Schrödinger equation:

$$
\begin{aligned}
i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(t-t_0)}(H-H_0)\,e^{-itH(t-t_0)} \\
&= e^{iH_0(t-t_0)}(H_{\text{int}})\,e^{-itH(t-t_0)} \\
&= \underbrace{e^{iH_0(t-t_0)}(H_{\text{int}})\,e^{-itH_0(t-t_0)}}\overbrace{e^{itH_0(t-t_0)}e^{-itH(t-t_0)}} \\
&= H_I(t)U(t,t_0)\,.
\end{aligned}
$$

- We expand $U \sim \exp(-iH_I t)$ as a power series in λ:

$$
\begin{aligned}
U(t,t_0) &= 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) \\
&\quad + \frac{(-i)^2}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \hat{T}\left[H_I(t_1)\,H_I(t_2)\right] + \cdots \\
&\equiv \hat{T}\left\{\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]\right\}.
\end{aligned}
$$

Interacting ground state

- The interacting ground state $|\Omega\rangle$ is not $|0\rangle$; $\langle\Omega|0\rangle \neq 0$.
- $E_0 \equiv \langle\Omega| H |\Omega\rangle$ with the zero of energy $H_0 |0\rangle = 0$.
- When $H |n\rangle = E_n |n\rangle$,

$$\begin{aligned} e^{-iHT} |0\rangle &= \sum_n e^{-iE_nT} |n\rangle \langle n|0\rangle \\ &= E^{-iE_0T} |\Omega\rangle \langle\Omega|0\rangle + \underbrace{\sum_{n\neq 0} e^{-iE_nT} |n\rangle \langle n|0\rangle}_{\to 0}. \end{aligned}$$

- Since $E_n > E_0$ for all $n \neq 0$, as we send $T \to \infty (1 - i\epsilon)$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T} \langle\Omega|0\rangle\right)^{-1} e^{-iHT} |0\rangle \,.$$

- Since T is very large, we can shift it $T \to \pm t_0$:

$$\begin{aligned} |\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0+T)} \langle\Omega|0\rangle\right)^{-1} U(t_0, -T) |0\rangle \\ \langle\Omega| &= \lim_{T\to\infty(1-i\epsilon)} \langle 0| U(T, t_0) \left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1}. \end{aligned}$$

Two-point correlation in the interacting system

The normalization of the interacting ground state is

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \langle 0| U(T, t_0) U(t_0, T) |0\rangle .$$

Now we have the two-point correlation function:

$$\begin{aligned} &\langle\Omega| \hat{T}\{\phi(x)\phi(y)\} |\Omega\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| \hat{T}\left\{\phi_I(x)\phi_I(y) \exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\} |0\rangle}{\langle 0| \hat{T}\left\{\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\} |0\rangle} \end{aligned}$$

We need to evaluate the expressions of the form

$$\langle 0| \hat{T}\{\phi_I(x_1) \phi_I(x_2) \cdots \phi_I(x_n)\} |0\rangle .$$

Note that $\langle 0| \hat{T}\{\phi_I(x)\phi_I(y)\} |0\rangle$ is just the Feynman propagtor.

Normal ordering

From now on we drop the subscript I.

We decompose $\phi(x)$ into the positive- and negative-frequency parts:

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ip\cdot x}, \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}{}^\dagger e^{+ip\cdot x}.$$

These decomposed fields satisfy

$$\phi^+(x)\left|0\right\rangle = 0 \quad \text{and} \quad \left\langle 0\right|\phi^-(x) = 0.$$

A normal ordring operator $\hat{N}$ is defined as

$$\hat{N}\left(a_{\mathbf{p}} a_{\mathbf{k}}{}^\dagger\right) \equiv a_{\mathbf{k}}{}^\dagger a_{\mathbf{p}} \quad \Rightarrow \quad \hat{N}\left\{\phi^+(x)\phi^-(y)\right\} = \phi^-(y)\phi^+(x).$$

So we, with $\left[a_{\mathbf{k}}{}^\dagger, a_{\mathbf{p}}\right]_\pm = a_{\mathbf{k}}{}^\dagger a_{\mathbf{p}} \pm a_{\mathbf{p}} a_{\mathbf{k}}{}^\dagger$, have identities

$$\begin{aligned} \left\langle 0\right| \hat{N}\left\{\phi(x)\phi(y)\right\}\left|0\right\rangle &= 0 \\ \hat{T}\left\{\phi(x)\phi(y)\right\} &= \hat{N}\left\{\phi(x)\phi(y)\right\} + \overbrace{\phi(x)\phi(y)}. \end{aligned}$$

Contractions and propagators

The *contraction* $\overline{ab}$ is defined by the commutators:

- The contraction for the Klein-Gordan fields is defined by

$$\overline{\phi(x)\phi(y)} \equiv \begin{cases} \left[\phi^+(x), \phi^-(y)\right], & \text{for } x^0 > y^0; \\ \left[\phi^+(y), \phi^-(x)\right], & \text{for } y^0 > x^0. \end{cases}$$

$$\langle 0|\hat{T}\left(\phi(x)\phi(y)\right)|0\rangle = \underbrace{\langle 0|\hat{N}\left(\phi(x)\phi(y)\right)|0\rangle}_{=0} + \langle 0|\overline{\phi(x)\phi(y)}|0\rangle$$

$$\overline{\phi(x)\phi(y)} = D_F(x-y) =$$

$$\overline{A^\mu(x)A^\nu(x)} = {\Delta_F}^{\mu\nu}(x-y) =$$ 

- The contraction for the Dirac field is defined by

$$\overline{\psi(x)\bar{\psi}(y)} \equiv \begin{cases} \left\{\psi^+(x), \bar{\psi}^-(y)\right\}, & \text{for } x^0 > y^0; \\ \left\{\bar{\psi}^+(y), \psi^-(x)\right\}, & \text{for } y^0 > x^0. \end{cases}$$

$$\langle 0|\hat{T}\left(\psi(x)\bar{\psi}(y)\right)|0\rangle = \langle 0|\hat{N}\left(\psi(x)\bar{\psi}(y)\right)|0\rangle + \langle 0|\overline{\psi(x)\bar{\psi}(y)}|0\rangle$$

$$\overline{\psi(x)\bar{\psi}(y)} = S_F(x-y) =$$ 

Wick's theorem and connected diagrams

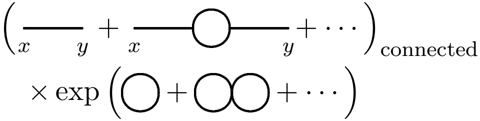
- For n field operators, we have an identity

$$\hat{T}\left[\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right] = \hat{N}\left[\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right] + \{\text{all possible contractions}\}\,.$$

  which is knwon as *Wick's theorem*.

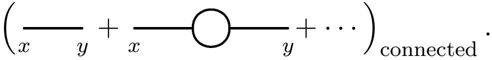
- The two-point correlation has the structure

$$\langle\Omega|\,\hat{T}\left\{\phi(x)\phi(y)\right\}|\Omega\rangle = \frac{\text{Numerator}}{\text{Denominator}},$$

Numerator =


Denominator =


$$\Downarrow$$

$$\langle\Omega|\,\hat{T}\left\{\phi(x)\phi(y)\right\}|\Omega\rangle =$$


S-matrix

S-matrix is simply the time-evolution operator, $\exp(-iHt)$:

$$\langle f, \mathrm{out}| S |i, \mathrm{out}\rangle = \lim_{T\to\infty} \langle f, \mathrm{out}| e^{-iH(2T)} |i, \mathrm{out}\rangle \,.$$

To compute this quantity we consider the external states ($|\Omega\rangle$):

$$|i, \mathrm{out}\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT} |i, 0\rangle \,.$$

The S-matrix will be of the form

$$\lim_{T\to\infty(1-i\epsilon)} \langle f, 0| e^{-iH(2T)} |i, 0\rangle$$

$$\propto \lim_{T\to\infty(1-i\epsilon)} \langle f, 0| \hat{T} \left(\exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right) |i, 0\rangle$$

Then the τ-matrix (*cf.*, $S = 1 + i\tau$) elements becomes

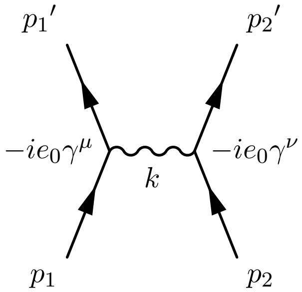
$$\langle f, \mathrm{out}| i\tau |i, \mathrm{out}\rangle = (2\pi)^4 \, \delta^{(4)} (P_i - P_f) \cdot i\mathcal{M} (i \to f)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(\langle f, 0| \hat{T} \left(\exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right) |i, 0\rangle \right)_{\substack{\text{connected} \\ \text{amputated}}} .$$

Coulomb interaction

The $\mathcal{M}$ matrix element of the Coulomb interaction in the leading order is

$$i\mathcal{M} =$$



$$= \bar{u}\left(p_1{}'\right)\left(-ie_0\gamma^\mu\right)u\left(p_1\right)\frac{-ig_{\mu\nu}}{k^2}\bar{u}\left(p_2{}'\right)\left(-ie_0\gamma^\nu\right)u\left(p_2\right)$$

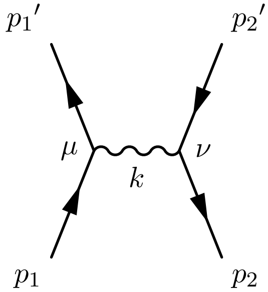
$$= \left(-ie_0\right)^2\bar{u}\left(p_1{}'\right)\gamma^\mu u\left(p_1\right)\frac{-ig_{\mu\nu}}{k^2}\bar{u}\left(p_2{}'\right)\gamma^\nu u\left(p_2\right).$$

This is known as the first part of the Møller scattering.

Bhabha scattering

The Bhabha scattering is a deformed Møller scattering:

$$i\mathcal{M} \quad =$$



$$= \quad \bar{u}\left(p_1{}'\right)\left(-ie_0\gamma^{\mu}\right)u\left(p_1\right)\frac{-ig_{\mu\nu}}{k^2}v\left(p_2{}'\right)\left(-ie_0\gamma^{\nu}\right)\bar{v}\left(p_2\right)$$

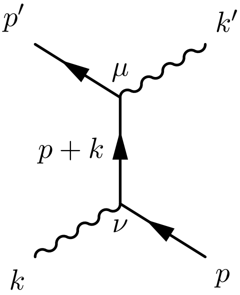
$$= \quad \left(-ie_0\right)^2\bar{u}\left(p_1{}'\right)\gamma^{\mu}u\left(p_1\right)\frac{-ig_{\mu\nu}}{k^2}v\left(p_2{}'\right)\gamma^{\nu}\bar{v}\left(p_2\right).$$

- The electrons-2 travels in reverse-time order T^{-1}.
- The CPT symmetry $\rightarrow (CP)^{-1}$.
- The negative-energy electron is known as *positron*.

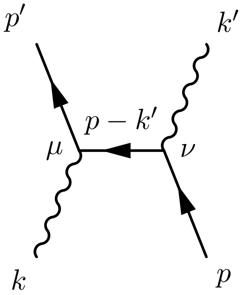
Compton scattering

The Compton scattering contains two diagrams,

$$i\mathcal{M} \quad =$$



$$+$$



$$= \bar{u}\left(p'\right)\epsilon_\mu{}^*\left(k'\right)\left(-ie_0\gamma^\mu\right)\frac{i\left(\not{p}+\not{k}+m_0\right)}{\left(p+k\right)^2-m_0{}^2}\left(-ie_0\gamma^\nu\right)\epsilon_\nu\left(k\right)u\left(p\right)$$

$$+\bar{u}\left(p'\right)\left(-ie_0\gamma^\nu\right)\epsilon_\nu\left(k\right)\frac{i\left(\not{p}-\not{k}'+m_0\right)}{\left(p-k\right)^2-m_0{}^2}\epsilon_\mu{}^*\left(k'\right)\left(-ie_0\gamma^\mu\right)u\left(p\right).$$

Compton scattering

The numerators and denominators can be simplified as follows:

- Since $p^2 = {m_0}^2$ and $k^2 = 0$, the denominators are

$$\begin{aligned} (p+k)^2 - {m_0}^2 &= 2p \cdot k \quad \text{in} \quad i\mathcal{M}_1 \\ \left(p-k'\right)^2 - {m_0}^2 &= -2p \cdot k \quad \text{in} \quad i\mathcal{M}_2. \end{aligned}$$

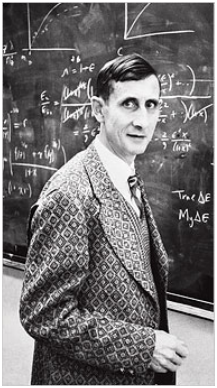
- For numerators, we use a bit of Dirac algebra:

$$\begin{aligned} \left(\not{p} + m_0\right) \gamma^\nu u(p) &= \left(2p^\nu - \gamma^\nu \not{p} + \gamma^\nu m_0\right) u(p) \\ &= 2p^\nu u(p) - \gamma^\nu \left(\not{p} - m_0\right) u(p) \\ &= 2p^\nu u(p), \end{aligned}$$

We obtain

$$\begin{aligned} i\mathcal{M} &= i{e_0}^2 {\epsilon_\mu}^* \left(k'\right) \epsilon_\nu\left(k\right) \\ &\quad \times \bar{u}\left(p'\right) \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{-\gamma^\nu \not{k}' \gamma^\mu + 2\gamma^\nu p^\nu}{-2p \cdot k'} \right] u(p). \end{aligned}$$

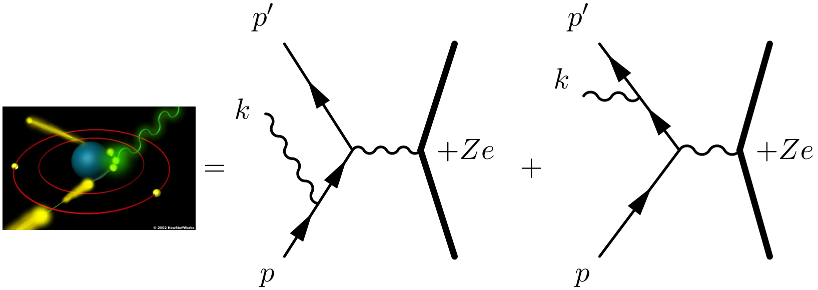
Renormalization



The laws of nature are constructed in such a way as to make the universe as interesting as possible. – Freeman Dyson.

Soft Bremsstrahlung

Bremsstrahlung = Bremsen (to break) + Strahlung (radiation).



For the soft photon radiation, $|\mathbf{k}| \ll |\mathbf{p}' - \mathbf{p}|$,

$$
\begin{aligned}
\mathcal{M}_0\left(p', p-k\right) &\approx \mathcal{M}_0\left(p'+k, p\right) \approx \mathcal{M}_0\left(p', p\right) \\
i\mathcal{M} &= -ie_0 \bar{u}\left(p'\right)\left[\mathcal{M}_0\left(p', p\right)\right] u(p) \left[e_0 \left(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right)\right]
\end{aligned}
$$

Soft Bremsstrahlung

The differential cross section is then

$$d\sigma\left(p \to p' + \gamma\right) = d\sigma\left(p \to p'\right) \times \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\sum_{\lambda=1,2} {e_0}^2 \left|\frac{p'\cdot\epsilon^{(\lambda)}}{p'\cdot k} - \frac{p\cdot\epsilon^{(\lambda)}}{p\cdot k}\right|^2 .$$

The differential probability becomes

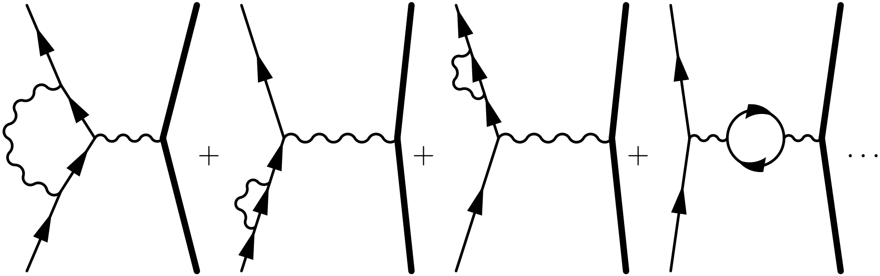
$$dP\left(p \to p' + \gamma(k)\right) = \frac{d^3k}{(2\pi)^3}\sum_{\lambda}\frac{{e_0}^2}{2k}\left|\boldsymbol{\epsilon}_\lambda\cdot\left(\frac{\mathbf{p}'}{p'\cdot k} - \frac{\mathbf{p}}{p\cdot k}\right)\right|^2 .$$

The total probability, for *soft* photons $0 \le k \le |\mathbf{q}| = |\mathbf{p}' - \mathbf{p}|$, and the differential cross section with fictitious photon mass μ, are

$$P \approx \int_0^{|\mathbf{q}|} dk \frac{1}{k}\mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) \sim \log\left(\frac{-q^2}{\mu^2}\right) \to \infty,$$

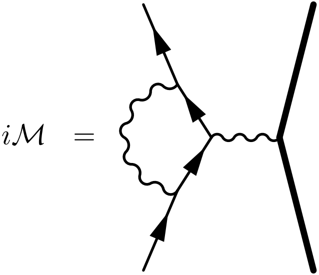
$$d\sigma\left(p \to p' + \gamma\right) \underset{-q^2\to\infty}{\propto} \frac{\alpha_0}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{{m_0}^2}\right).$$

Radiative corrections



There are three classes of the radiative corrections;
Vertex corrections, Self-energies, and Polarizations.

Vertex corrections



$$i\mathcal{M}(2\pi)\,\delta\left(p'^0 - p^0\right) = -ie_0\left(\bar{u}\left(p'\right)\Gamma^\mu\left(p', p\right)u(p)\right)\cdot\tilde{A}^{\rm cl}_\mu\left(p' - p\right).$$

To lowest order, $\Gamma^\mu = \gamma^\mu$.
We may express Γ^μ in a symmetrical form:

$$\Gamma^\mu = \gamma^\mu A + \left(p'^\mu + p^\mu\right)B + \left(p'^\mu - p^\mu\right)C.$$

Vertex corrections

Using the Gordon identity

$$\bar{u}\left(p'\right)\gamma^{\mu}u(p) = \bar{u}\left(p'\right)\left[\frac{p'^{\mu}+p^{\mu}}{2m_0} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m_0}\right]u(p),$$

we have

$$\Gamma^{\mu}\left(p',p\right) = \gamma^{\mu}F_1\left(q^2\right) + i\frac{\sigma^{\mu\nu}q_{\nu}}{2m_0}F_2\left(q^2\right),$$

where the unknown functions F_1 and F_2 are called *form factors*. To lowest order, $F_1 = 1$ and $F_2 = 0$.

- When $\tilde{A}^{\text{cl}}_{\mu}(x) = \left((2\pi)\,\delta\left(q^0\right)\phi\left(\mathbf{q}\right),\mathbf{0}\right)$,

$$\begin{aligned} i\mathcal{M} &= -ie_0\bar{u}\left(p'\right)\Gamma^{0}\left(p',p\right)u(p)\cdot\tilde{\phi}\left(\mathbf{q}\right) \\ &\underset{\mathbf{q}\to 0}{=} -ie_0F_1(0)\tilde{\phi}\left(\mathbf{q}\right)\cdot 2m_0{\xi'}^{\dagger}\xi \\ V\left(\mathbf{x}\right) &= e_0F_1(0)\phi\left(\mathbf{x}\right). \end{aligned}$$

Vertex corrections

- When $A_\mu^{\text{cl}}(x) = \left(0, \mathbf{A}^{\text{cl}}(\mathbf{x})\right)$,

$$i\mathcal{M} = +ie_0 \left[\bar{u}\left(p'\right)\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m_0}F_2\right)u(p)\right]\tilde{A}^i_{\text{cl}}\left(\mathbf{q}\right).$$

Again the expression in brackets vanishes at $\mathbf{q} = 0$, so in this limit

$$i\mathcal{M} = -i\left(2m_0\right)\cdot e_0 \xi'^\dagger \left(\frac{-1}{2m_0}\sigma^k\left[F_1(0) + F_2(0)\right]\right)\xi \tilde{B}^k\left(\mathbf{q}\right),$$

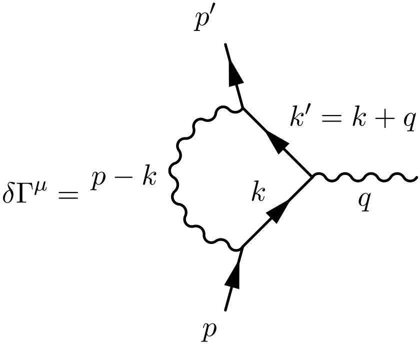
where $\tilde{B}^k\left(\mathbf{q}\right) = -i\epsilon^{ijk}q^i\tilde{A}^j_{\text{cl}}\left(\mathbf{q}\right)$. This is just that of a magnetic moment interaction $V\left(\mathbf{x}\right) = -\left\langle\boldsymbol{\mu}\right\rangle\cdot\mathbf{B}\left(\mathbf{x}\right)$, where

$$\begin{aligned}\left\langle\boldsymbol{\mu}\right\rangle &= \frac{e_0}{m_0}\left[F_1(0) + F_2(0)\right]\xi'^\dagger\frac{\boldsymbol{\sigma}}{2}\xi, \\ \boldsymbol{\mu} &= g\left(\frac{e_0}{2m_0}\right)\mathbf{S},\end{aligned}$$

where the *Landé g-factor* is $g = 2\left[F_1(0) + F_2(0)\right] = 2 + 2F_2(0)$.

Vertex corrections

To one-loop order, the vertex function $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$:



where

$$\bar{u}\left(p'\right)\delta\Gamma^\mu\left(p',p\right)u(p) =$$

$$2ie_0{}^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}\left(p'\right)\left[\not{k}\gamma^\mu\not{k}' + m_0{}^2\gamma^\mu - 2m_0\left(k+k'\right)^\mu\right]u(p)}{\left((k-p)^2+i\epsilon\right)\left(k'^2-m_0{}^2+i\epsilon\right)\left(k^2-m_0{}^2+i\epsilon\right)}.$$

Vertex corrections

The Feynman parameterization

$$\begin{aligned}\frac{1}{AB} &= \int_0^1 dx \frac{1}{\left[xA+(1-x)B\right]^2} \\ &= \int_0^1 dx dy \delta\left(x+y-1\right)\frac{1}{\left[xA+yB\right]^2}\end{aligned}$$

simplifies the denominator D to

$$\begin{aligned}D &= l^2-\Delta+i\epsilon \\ l &\equiv k+yq-zp, \\ \Delta &\equiv -xyq^2+(1-z)^2 {m_0}^2 > 0.\end{aligned}$$

The numerator will be

$$N=\bar{u}\left(p'\right)\left[\begin{array}{r}\gamma^\mu\cdot\left(-\frac{1}{2}l^2+(1-x)(1-y)q^2+\left(1-2z-z^2\right){m_0}^2\right) \\ +\left(p'^\mu+p^\mu\right)\cdot m_0 z(z-1) \\ +q^\mu\cdot m_0(z-2)(x-y)\end{array}\right]u(p).$$

Vertex corrections

Using the Gordon identity again, we have an entire expression

$$
\begin{aligned}
&\bar{u}\left(p'\right)\delta\Gamma^{\mu}\left(p',p\right)u(p) = \\
&\quad 2ie_0{}^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta\left(x+y+z-1\right)\frac{2}{D^3} \\
&\quad\times\bar{u}\left(p'\right)\left[\gamma^{\mu}\cdot\left(-\frac{1}{2}l^2+(1-x)(1-y)q^2+\left(1-4z+z^2\right)m_0{}^2\right)\right. \\
&\qquad\left.+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m_0}\left(2m_0{}^2z(1-z)\right)\right]u(p).
\end{aligned}
$$

There are two classes of integrations:

$$
\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n}}_{\text{convergent}}\quad\text{and}\quad\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^n}}_{\text{divergent for } n\leq 3}.
$$

Pauli-Villars regularizatoin

We introduce *ad hoc* a cut-off $\Lambda(\to \infty)$ in the photon propagators:

$$\frac{1}{(k^2-p^2)+i\epsilon} \to \frac{1}{(k^2-p^2)+i\epsilon} - \frac{1}{(k^2-p^2)-\Lambda^2+i\epsilon}$$

so the denominator is altered as

$$\Delta \to \Delta_\Lambda = -xyq^2 + (1-z)^2{m_0}^2 + z\Lambda^2.$$

Then the divergent integral is replaced by convergent ones:

$$\int \frac{d^4l}{(2\pi)^4}\left(\frac{l^2}{(l^2-\Delta)^3} - \frac{l^2}{(l^2-\Delta_\Lambda)^3}\right) = \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right),$$

which looks like $(\infty - \infty_\Lambda) \propto \log(\Delta_\Lambda/\Delta)$.

- How can this result affect on $F_1(q^2)$ and $F_2(q^2)$ with $F_1(0)=1$?

The convergent form factor F_2

The form factor F_2 is corrected to order $\alpha_0 (= {e_0}^2/4\pi\hbar c)$

$$F_2\left(q^2\right) = \frac{\alpha_0}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left[\frac{2{m_0}^2 z(1-z)}{{m_0}^2(1-z)^2 - q^2 xy}\right]$$

is convergent especially for $q^2 = 0$, such that

$$\begin{aligned} F_2\left(q^2=0\right) &= \frac{\alpha_0}{2\pi}\int_0^1 dxdydz\delta\left(x+y+z-1\right)\frac{2{m_0}^2 z\left(1-z\right)}{{m_0}^2\left(1-z\right)^2} \\ &= \frac{\alpha_0}{\pi}\int_0^1 dz\int_0^{1-z} dy\frac{z}{1-z} = \frac{\alpha_0}{2\pi}. \end{aligned}$$

We get a correction to the g-factor of the electron:

$$a_e \equiv \frac{g-2}{2} = \frac{\alpha_0}{2\pi} \underset{(\alpha_0 \hat{=} \alpha)}{\approx} 0.0011614.$$

Experiments give ${a_e}^{\text{exp}} = 0.0011597$, which differs by ≈ 0.15 %.

Infrared divergence

The divergent form factor $F_1\left(q^2\right)$ is corrected to

$$
\begin{aligned}
F_1\left(q^2\right) &= 1+\frac{\alpha_0}{2\pi}\int_0^1 dxdydz\delta\left(x+y+z-1\right) \\
&\times\left[\log\left(\frac{{m_0}^2\left(1-z\right)^2}{{m_0}^2\left(1-z\right)^2-q^2xy}\right)\right. \\
&\quad +\frac{{m_0}^2\left(1-4z+z^2\right)+q^2\left(1-x\right)\left(1-y\right)}{{m_0}^2\left(1-z\right)^2-q^2xy+\mu^2 z} \\
&\quad \left.-\frac{{m_0}^2\left(1-4z+z^2\right)}{{m_0}^2\left(1-z\right)^2+\mu^2 z}\right],
\end{aligned}
$$

where μ is the fictitious photon mass.
In the limit $\mu\rightarrow 0$, we may obtain

$$
F_1\left(-q^2\rightarrow\infty\right)=1-\frac{\alpha_0}{2\pi}\log\left(\frac{-q^2}{{m_0}^2}\right)\log\left(\frac{-q^2}{\mu^2}\right).
$$

What did we have made mistake?

- The S-matrix theory

$$\langle\Omega|\hat{T}\phi(x_1)\phi(x_2)\cdots|\Omega\rangle = \sum \begin{pmatrix} \text{connected} \\ \text{amputated} \end{pmatrix}$$

  is based on the completeness of the *normalized* interacting ground state $|\Omega\rangle$:

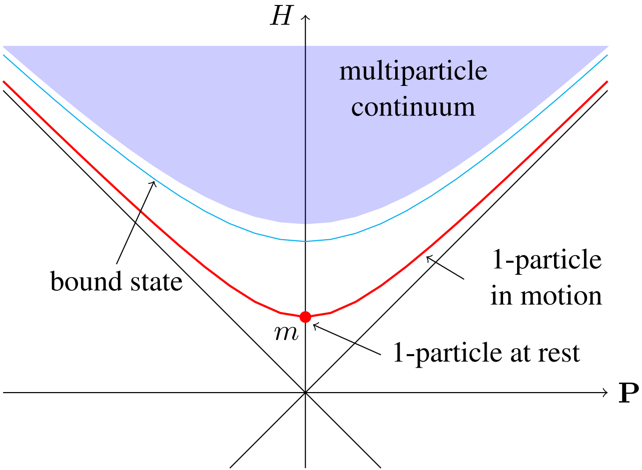
$$\mathbf{1} = |\Omega\rangle\langle\Omega|$$

  from the free vacuum $|0\rangle$.
- Let $H|\lambda_0\rangle = \lambda_0|\lambda_0\rangle$, but $\mathbf{P}|\lambda_0\rangle = 0$.
- Let $|\lambda_\mathbf{p}\rangle$ be the *boosts* of $|\lambda_0\rangle$ with $E_\mathbf{p}(\lambda) \equiv \sqrt{|\mathbf{p}|^2 + m_\lambda{}^2}$.
- The desired completeness relation will be

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}(\lambda)}|\lambda_\mathbf{p}\rangle\langle\lambda_\mathbf{p}|.$$

- Accordingly we need to normalize $|\Omega\rangle$ *again*:
  $\Rightarrow$***Renormalization***.

The particle dispersion



The eigenvalues of $P^\mu = (H, \mathbf{P})$ of particle mass m.

Renormalization

Assume $x^0 > y^0$ and drop off $\langle\Omega|\,\phi(x)\,|\Omega\rangle\,\langle\Omega|\,\phi(y)\,|\Omega\rangle\;(=0)$.
The two-point correlation function is

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}(\lambda)}\,\langle\Omega|\,\phi(x)\,|\lambda_{\mathbf{p}}\rangle\,\langle\lambda_{\mathbf{p}}|\,\phi(y)\,|\Omega\rangle\,.$$

The matrix element

$$\begin{aligned}\langle\Omega|\,\phi(x)\,|\lambda_{\mathbf{p}}\rangle &= \langle\Omega|\,e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\,|\lambda_{\mathbf{p}}\rangle \\ &= \langle\Omega|\,\phi(0)\,|\lambda_{\mathbf{p}}\rangle\; e^{-ip\cdot x}\Big|_{p^0=E_{\mathbf{p}}} \\ &= \langle\Omega|\,\phi(0)\,|\lambda_0\rangle\; e^{-ip\cdot x}\Big|_{p^0=E_{\mathbf{p}}}\,.\end{aligned}$$

The two-point correlation function becomes for $x^0 > y^0$

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-{m_\lambda}^2+i\epsilon}e^{-ip\cdot(x-y)}\,\left|\langle\Omega|\,\phi(0)\,|\lambda_0\rangle\right|^2\,.$$

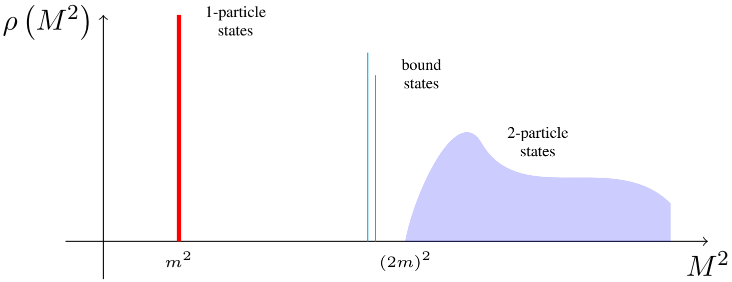
Källén-Lehmann representation

For both cases of $x^0 > y^0$ and $y^0 > x^0$, we have

$$\langle\Omega|\,\hat{T}\phi(x)\phi(y)\,|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right)D_F\left(x-y;M^2\right),$$

where $\rho\left(M^2\right)$ is a positive *spectral density*,

$$\rho\left(M^2\right) = \sum_\lambda (2\pi)\,\delta\left(M^2 - {m_\lambda}^2\right)\left|\langle\Omega|\,\phi(0)\,|\lambda_0\rangle\right|^2.$$



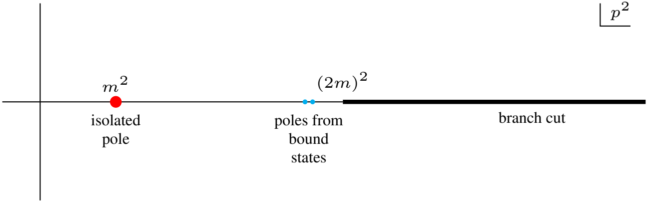
Field-strength renormalization

The spectral density is

$$\rho\left(M^2\right) = 2\pi\delta\left(M^2 - m^2\right) Z + \left(\text{nothing else for } M^2 \lesssim (2m)^2\right),$$

where Z is refered as *field-strength renormalization*.
The Fourier transform of the two-point correlation becomes

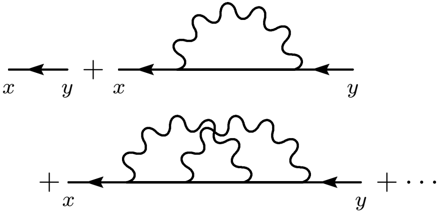
$$\int d^4x e^{ip\cdot x} \langle\Omega|\,\hat{T}\phi(x)\phi(0)\,|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right)\frac{i}{p^2 - M^2 + i\epsilon}$$
$$= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim(2m)^2}^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right)\frac{i}{p^2 - M^2 + i\epsilon}.$$



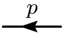
The electron self-energy

- The electron two-point correlation function is

$$\langle\Omega|\hat{T}\psi(x)\bar{\psi}(y)|\Omega\rangle =$$

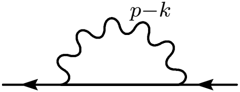


- The free-field propagator:



$$= \frac{i\left(\not{p}+m_0\right)}{p^2-{m_0}^2+i\epsilon}.$$

- The lowest order *electron self-energy*:



$$= \frac{i\left(\not{p}+m_0\right)}{p^2-{m_0}^2}\left[-i\Sigma_2\left(p\right)\right]\frac{i\left(\not{p}+m_0\right)}{p^2-{m_0}^2}.$$

The electron self-energy

We have the explicit form of the electron self-energy:

$$-i\Sigma_2(p) = (-ie_0)^2 \int \frac{d^4k}{(2\pi)^2} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0{}^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon},$$

where we regulate it by adding a small *photon mass* μ.
We use the Feynman parametrization and shift the momentum $l = k - xp$ to get

$$-i\Sigma_2(p) = -e_0{}^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^3} \frac{-2x\not{p} + 4m_0}{(l^2 - \Delta + i\epsilon)^2},$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0{}^2$.
We regulate it by the Pauli-Villars procedure:

$$\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} \to \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} - \frac{1}{(p-k)^2 - \Lambda^2 + i\epsilon}.$$

The electron self-energy

Introducing $\Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2(1-x){m_0}^2 \underset{\Lambda\to\infty}{\to} x\Lambda^2$,
we have

$$\Sigma_2(p) = \frac{\alpha_0}{2\pi}\int_0^1 dx\,\left(2m_0 - x\not{p}\right)\log\left(\frac{x\Lambda^2}{(1-x){m_0}^2 + x\mu^2 - x(1-x)p^2}\right).$$

The logarithm of x has a branch cut begining at the point where

$$(1-x)\,{m_0}^2 + x\mu^2 - x\,(1-x) = 0,$$

or

$$x = \frac{1}{2} + \frac{{m_0}^2}{2p^2} - \frac{\mu^2}{2p^2} \pm \frac{1}{2p^2}\sqrt{\left(p^2 - (m_0+\mu)^2\right)\left(p^2 - (m_0-\mu)^2\right)}.$$

The branch cut of $\Sigma_2\left(p^2\right)$ begins at $p^2 = (m_0+\mu)^2$,
two-particle threshold.

- Where is the simple pole at $p^2 = m^2$?

The electron self-energy

The two-point correlation function is written as

$$
\begin{aligned}
&= \frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2} + \frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2}\left(-i\Sigma\right)\frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2} \\
&\quad + \frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2}\left(-i\Sigma\right)\frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2}\left(-i\Sigma\right)\frac{i\left(\not p + m_0\right)}{p^2 + {m_0}^2} + \cdots \\
&= \frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\left(\frac{\Sigma\left(\not p\right)}{\not p - m_0}\right) + \frac{i}{\not p - m_0}\left(\frac{\Sigma\left(\not p\right)}{\not p - m_0}\right)^2 + \cdots \\
&= \frac{i}{\not p - m_0 - \Sigma\left(\not p\right)}.
\end{aligned}
$$

Hence $$\left(\not p - m_0 - \Sigma\left(\not p\right)\right)\Big|_{\not p = m} = 0$$

gives us the simple pole at the *physical mass*, $m = m_0 + \Sigma\left(\not p\right)$.

Mass renormalization

In the vicinity of the pole, $\not{p} - m_0 - \Sigma\left(\not{p}\right)$ has the form

$$\left(\not{p} - m\right) \cdot \left(1 - \left.\frac{d\Sigma\left(\not{p}\right)}{d\not{p}}\right|_{\not{p}=m}\right) + \mathcal{O}\left(\left(\not{p} - m\right)^2\right).$$

When we write the two-point correlation function as

$$\int d^4x e^{ip\cdot x} \left\langle\Omega\right| \hat{T}\psi(x)\bar{\psi}(0) \left|\Omega\right\rangle = \frac{iZ_2\left(\not{p} + m\right)}{p^2 - m^2 + i\epsilon},$$

we obtain the *mass renormalization constant* to be

$${Z_2}^{-1} = 1 - \left.\frac{d\Sigma\left(\not{p}\right)}{d\not{p}}\right|_{\not{p}=m}.$$

Mass renormalization

To order α_0, the mass shift is

$$\delta m = m - m_0 = \Sigma_2\left(\not p = m\right) \approx \Sigma_2\left(\not p = m_0\right).$$

Then the mass shift is

$$\delta m = \frac{\alpha_0}{2\pi} m_0 \int_0^1 dx\,(1-x)\log\left(\frac{x\Lambda^2}{(1-x)^2 {m_0}^2 + x\mu^2}\right),$$

which is ultraviolet divergent $\mathcal{O}\left(\log\Lambda^2\right)$ for $\Lambda \to \infty$.
The correction for Z_2 in order α_0 is calculated to be

$$\begin{aligned} \delta Z_2 &= \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p = m} \\ &= \frac{\alpha_0}{2\pi}\int_0^1 dx \left[-x\log\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right. \\ &\qquad \left. + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2}\right]. \end{aligned}$$

¶The *small* correction to mass m_0 is *infinite*!

Mass renormalizaiton

One can show that the *exact* vertex should be read

$$Z_2\Gamma^\mu\left(p', p\right) = \gamma^\mu F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2\left(q^2\right).$$

The left-hand side of the exact vertex function becomes

$$Z_2\Gamma^\mu = \left(1+\delta Z_2\right)\left(\gamma^\mu + \delta\Gamma^\mu\right) = \gamma^\mu + \delta\Gamma^\mu + \gamma^\mu \delta Z_2,$$

while in the right-hand side $F_1\left(q^2\right)$ becomes

$$F_1\left(q^2\right) = 1 + \delta F_1\left(q^2\right) + \delta Z_2 = 1 + \left[\delta F_1\left(q^2\right) - \delta F_1\left(0\right)\right],$$

if $\delta Z_2 = -\delta F_1(0)$.
Define another rescaling factor Z_1 by the relation

$$\Gamma^\mu\left(q=0\right) = {Z_1}^{-1}\gamma^\mu,$$

where Γ^μ is the complete amputated vertex function.

Mass renormalization

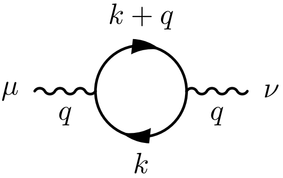
However, the divergent part of the vertex correction is

$$\begin{aligned}\delta F_1(0) &= \frac{\alpha_0}{2\pi}\int_0^1 dxdydz\delta\left(x+y+z-1\right)\\ &\quad\times\left[\log\left(\frac{z\Lambda^2}{\left(1-z\right)^2 m^2+z\mu^2}\right)+\frac{\left(1-4z+z^2\right)m^2}{\left(1-z\right)^2 m^2+z\mu^2}\right]\\ &= \frac{\alpha_0}{2\pi}\int_0^1 dz\left(1-z\right)\\ &\quad\times\left[\log\left(\frac{z\Lambda^2}{\left(1-z\right)^2 m^2+z\mu^2}\right)+\frac{\left(1-4z+z^2\right)m^2}{\left(1-z\right)^2 m^2+z\mu^2}\right].\end{aligned}$$

We can show that $\delta F_1(0)+\delta Z_2=0$.
To find $F_1(0)=1$, we must provide the identity $Z_1=Z_2$,
so that the vertex rescaling *exactly* compensates the electron field-strength renormalization.
¶The understanding of mass is postponed.

Vacuum polarization

Photon is dressed in order ${e_0}^2$

$i{\Pi_2}^{\mu\nu}(q) \equiv$ 

$$= (-1)(-ie_0)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \frac{i}{\not k - m} \gamma^\nu \frac{i}{\not k + \not q - m}\right].$$

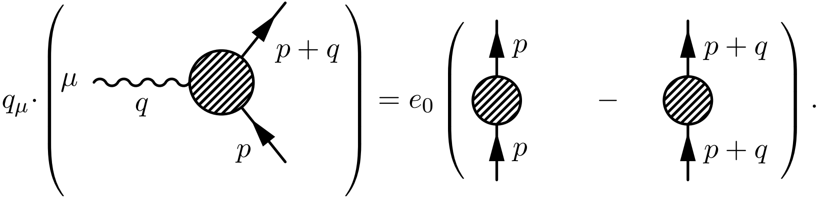
Generally the polarized photon propagator is defined by

$i\Pi^{\mu\nu}(q) \equiv$ 

As for the electron self-energy the polarization decomposes

Ward-Takahashi identity

The guage invariance of radiation field leads the charge conservation ($q_\mu \mathcal{M}^\mu(q) = 0$) in such a way that



This identity is known as the *Ward-Takahashi identity*:

$$-iq_\mu \Gamma^\mu (p+q, p) = S^{-1}(p+q) - S^{-1}(p)\,.$$

We defined Z_1 and Z_2 by the relations

$$\Gamma^\mu (p+q, p) \to {Z_1}^{-1}\gamma^\mu \text{ as } q \to 0 \text{ and } S(p) \sim \frac{iZ_2}{\not{p} - m}.$$

Setting p near mass shell and expanding the Ward-Takahashi identity about $q = 0$, we find

$$-i{Z_1}^{-1}\not{q} = -i{Z_2}^{-1}\not{q} \Rightarrow Z_1 = Z_2.$$

Charge renormalization

- The Ward-Takahashi identity tells us that $q_\mu \Pi^{\mu\nu} = 0$.
- In other words, $\Pi^{\mu\nu} \propto \left(g^{\mu\nu} - q^\mu q^\nu / q^2\right)$.
- Furthermore, we can expect $\Pi^{\mu\nu}(q)$ will not have a pole at $q^2 = 0$.
- It is convenient to write

$$\Pi^{\mu\nu}\left(q\right) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi\left(q^2\right),$$

where $\Pi\left(q^2\right)$ is regular at $q^2 = 0$.

The exact photon two-point correlation function is

$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\nu}}{q^2}\left[i\left(q^2 g^{\rho\sigma} - q^\rho q^\sigma\right)\Pi\left(q\right)\right]\frac{-i g_{\sigma\nu}}{q^2} + \cdots$$

$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{-i g_{\mu\rho}}{q^2}\Delta^\rho_\nu \Pi\left(q^2\right) + \frac{-i g_{\mu\rho}}{q^2}\Delta^\rho_\sigma \Delta^\sigma_\nu \Pi^2\left(q^2\right) + \cdots$$

where $\Delta^\rho_\nu \equiv \delta^\rho_\nu - q^\rho q_\nu / q^2$ and $\Delta^\rho_\sigma \Delta^\sigma_\nu = \Delta^\rho_\nu$.

Charge renormalization

- We can simplify further

$$\begin{aligned} &= \frac{-i}{q^2\left(1-\Pi\left(q^2\right)\right)}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+\frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right) \\ &= \frac{-ig_{\mu\nu}}{q^2\left(1-\Pi\left(q^2\right)\right)}\left(\because q_\mu \mathcal{M}^\mu(q)=0\right). \end{aligned}$$

- As long as $\Pi\left(q^2\right)$ is regular at $q^2=0$, the exact propagator alway has a pole at $q^2=0$.
- In other words, the photon remains absolutely *massless* at all orders in the perturbation theory.
- The residue of the $q^2=0$ pole is

$$\frac{1}{1-\Pi(0)} \equiv Z_3.$$

Charge renormalization

- Since the scattering amplitude will be shifted by

$$\cdots \frac{{e_0}^2 g_{\mu\nu}}{q^2} \cdots \to \cdots \frac{Z_3 {e_0}^2 g_{\mu\nu}}{q^2} \cdots ,$$

  we will have the *charge renormalization*

$$e = \sqrt{Z_3} e_0 .$$

- Considering a scattering process with nonzero q^2 in leading order α_0,

$$\frac{i g_{\mu\nu}}{q^2} \left(\frac{{e_0}^2}{1 - \Pi\left(q^2\right)} \right) \approx \frac{-i g_{\mu\nu}}{q^2} \left(\frac{{e_0}^2}{1 - \left[\Pi_2\left(q^2\right) - \Pi_2\left(0\right)\right]} \right) .$$

- The quantity in $(\cdots)$ has an interpretation of a q^2-dependent electric charge, so we have

$$\alpha_0 \to \alpha\left(q^2\right) = \frac{{e_0}^2/4\pi}{1 - \Pi\left(q^2\right)} \approx \frac{\alpha_0}{1 - \left[\Pi_2\left(q^2\right) - \Pi_2\left(0\right)\right]} .$$

The divergent polarization Π_2

In order ${e_0}^2$, the polarization is *badly* ultraviolet divergent:

$$i{\Pi_2}^{\mu\nu}(q) = -\left(-ie_0\right)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \frac{i\left(\not{k}+m\right)}{k^2-m^2}\gamma^\nu \frac{i\left(\not{k}+\not{q}+m\right)}{(k+q)^2-m^2}\right]$$

$$= -4{e_0}^2 \int \frac{d^4k}{(2\pi)^3} \frac{k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}\left(k\cdot(k+q)-m^2\right)}{(k^2-m^2)\left((k+q)^2-m^2\right)}.$$

Introducing a Feynman parameter, we combine the denominator as

$$\frac{1}{(k^2-m^2)\left((k+q)^2-m^2\right)} = \int_0^1 dx \frac{1}{\left(l^2+x(1-x)q^2-m^2\right)^2},$$

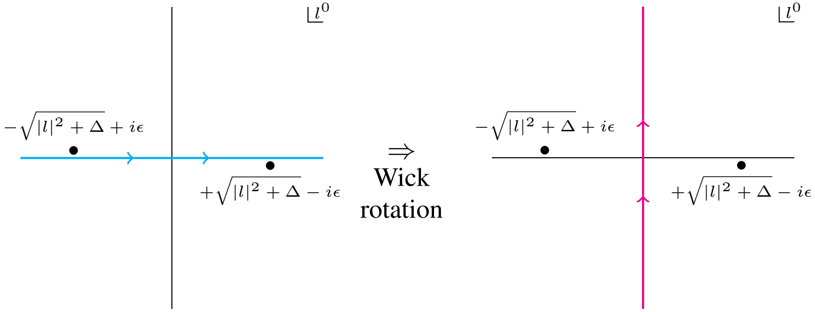
where $l \equiv k + xq$. In terms of l, the numerator will be

$$\begin{aligned}\text{Numerator} &= 2l^\mu l^\nu - g^{\mu\nu} l^2 - 2x(1-x)q^\mu q^\nu \\ &\quad + + g^{\mu\nu}\left(m^2 + x(1-x)q^2\right) + (\text{terms linear in } l)\,.\end{aligned}$$

Wick rotation

- The momentum (contour) integral in the Minkowski metric space-time, $g^{\mu\mu} = (+1, -1, -1, -1)$, is difficult.
- So Wick suggested a rotation of the time coordinate $t \to -ix^0$, i.e., the Euclidean four-vector product:

$$x^2 = t^2 - |\mathbf{x}|^2 \to -\left(x^0\right)^2 - |\mathbf{x}|^2 = -\left|x_E\right|^2 .$$



Dimensional regularization

- For sufficiently small dimension d, any loop-momentum integral will converge.
- Therefore the Ward identity can be proved.
- The final expression for Π_2 should have well-defined limit as $d \to 4$.
- A typical d-dimensional Euclidean space integral reads

$$\int \frac{d^d l_E}{(2\pi)^2} \frac{1}{\left({l_E}^2 + \Delta\right)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \cdot \int_0^\infty dl_E \frac{{l_E}^{d-1}}{\left({l_E}^2 + \Delta\right)^2},$$

where the area of a unit sphere in d dimensions is identified as

$$\int d\Omega_d = \frac{2\left(\sqrt{\pi}\right)^d}{\Gamma\left(\frac{d}{2}\right)}$$

and the second factor of the integral becomes

$$\int_0^\infty dl_E \frac{l^{d-1}}{\left({l_E}^2 + \Delta\right)^2} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)}.$$

Dimensional regularization

- Near $d = 4$, define $\epsilon = 4 - d$, and use the approximation

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon),$$

  where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.
- The integral is then

$$\int \frac{d^4 l_E}{(2\pi)} \frac{1}{\left({l_E}^2 + \Delta\right)^2} \underset{\epsilon \to 0}{\to} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\right)$$

- In d dimensions, $g_{\mu\nu}g^{\mu\nu} = d$.
- Thus, $l^\mu l^\nu$ of the numerators in the integrands should be replaced by $\frac{1}{d} l^2 g^{\mu\nu}$.
- The Dirac matrices in $d = 4 - \epsilon$ should be modified to

$$\begin{aligned} \gamma^\mu \gamma^\nu \gamma_\mu &= -(2 - \epsilon)\gamma^\nu \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho \\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma. \end{aligned}$$

Evaluation of Π_2

The unpleasant terms with l^2 in the numerator gives

$$\int \frac{d^d l_E}{(2\pi)^2} \frac{\left(-\frac{2}{d}+1\right) g^{\mu\nu} {l_E}^2}{\left({l_E}^2+\Delta\right)^2} = \frac{1}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \left(-\Delta g^{\mu\nu}\right).$$

Evaluating remaining terms and using $\Delta = m^2 - x\left(1-x\right) q^2$ are

$$i{\Pi_2}^{\mu\nu}\left(q\right) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot i\Pi_2\left(q^2\right),$$

where

$$\begin{aligned} \Pi_2\left(q^2\right) &= -\frac{8{e_0}^2}{(4\pi)^{d/2}} \int_0^1 dx x\left(1-x\right) \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \\ &\underset{\epsilon\to 0}{\to} -\frac{2\alpha_0}{\pi} \int_0^1 dx x\left(1-x\right) \left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log\left(4\pi\right)\right). \end{aligned}$$

This satisfies the Ward identity,
but it is still logarithmically *divergent*.

The electron charge shift

- In order α_0 the electric charge shift is computed as

$$\frac{e^2 - {e_0}^2}{{e_0}^2} = \delta Z_3 \to \Pi_2(0) \approx -\frac{2\alpha_0}{3\pi\epsilon} \underset{\epsilon\to 0}{\to} \infty.$$

- The *bare* charge is infinitely larger than the observed charge.
- This bare charge is *not* observable.
- What can be observed is

$$\alpha\left(q^2\right) \approx \frac{\alpha_0}{1 - \left[\Pi_2\left(q^2\right) - \Pi_2(0)\right]} \equiv \frac{\alpha_0}{1 - \hat{\Pi}_2\left(q^2\right)},$$

where the difference

$$\hat{\Pi}_2\left(q^2\right) = -\frac{2\alpha_0}{\pi}\int_0^1 dx x\left(1-x\right)\log\left(\frac{m^2}{m^2 - x\left(1-x\right)q^2}\right),$$

which is independent of ϵ in the limit $\epsilon \to 0$.

Classical picture

In nonrelativistic limit, the attractive Coulomb potential reads

$$V(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{-e^2}{|\mathbf{q}|^2 \left(1 - \hat{\Pi}_2\left(-|\mathbf{q}|^2\right)\right)}.$$

Expanding $\hat{\Pi}_2$ for $\left|q^2\right| \ll m^2$, we obtain

$$\begin{aligned} V(\mathbf{x}) &= -\frac{\alpha}{r} - \frac{4\alpha^2}{15m^2}\delta^{(3)}(\mathbf{x}) \\ &= \frac{ie^2}{(2\pi)^2}\left(\frac{1}{r}\right)\int_{-\infty}^{\infty} dQ \frac{Qe^{iQr}}{Q^2+\mu^2}\left(1 + \hat{\Pi}_2\left(-Q^2\right)\right). \end{aligned}$$

When $r^{-1} \gg m (= \lambda_C)$, we can approximate the potential as

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \cdots\right)$$

$\rightarrow$ *vacuum polarizations*–virtual dipoles screening.

Short-distance limit

For small distance or $-q^2 \gg m^2$, we have

$$\begin{aligned}
\hat{\Pi}_2\left(q^2\right) &\approx \frac{2\alpha}{\pi}\int_0^1 dx x\left(1-x\right) \\
&\quad \times\left[\log\left(\frac{-q^2}{m^2}\right)+\log\left(x\left(1-x\right)\right)+\mathcal{O}\left(\frac{m^2}{q^2}\right)\right] \\
&= \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right)-\frac{5}{3}+\mathcal{O}\left(\frac{m^2}{q^2}\right)\right].
\end{aligned}$$

The effective coupling constant in this limit is therefore

$$\alpha_{\text{eff}}\left(q^2\right)=\frac{\alpha}{1-\frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)},\quad A=\exp\left(5/3\right).$$

The effective electric charge becomes much larger at small distances, as we penetrate the screening cloud of virtual electron-positron pairs.

Renormalized quantum electrodynamics

The original QED Lagrangian is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e_0\bar{\psi}\gamma^\mu\psi A_\mu .$$

The renormalization scheme modifies the electron and photon propagators as

$$= \frac{iZ_2}{p\!\!\!/ - m} + \cdots$$

$$= \frac{-iZ_3 g_{\mu\nu}}{q^2} + \cdots$$

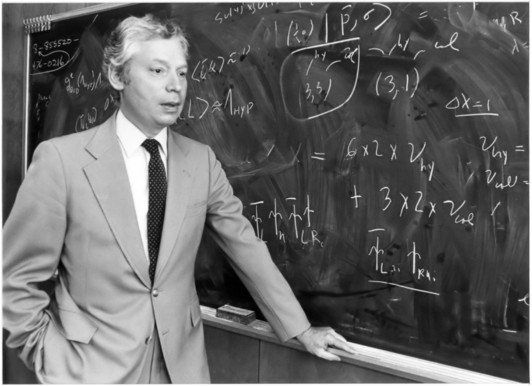
To absorb Z_2 and Z_3 into $\mathcal{L}$, we substitute $\psi \to {Z_2}^{1/2}\psi$ and $A^\mu \to {Z_3}^{1/2}A^\mu$. The Lagrangian becomes

$$\mathcal{L} = Z_2\bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - \frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} - e_0 Z_2 {Z_3}^{1/2}\bar{\psi}\gamma^\mu\psi A_\mu ,$$

with the physical electric charge

$$e_0 Z_2 {Z_3}^{1/2} = eZ_1 .$$

Renormalization Group and Higgs mechanism



It seems that scientists are often attracted to beautiful theories in the way that insects are attracted to flowers. – Steven Weinberg.

Cutoff problem

- Our coupling constant (fine-structure constant) is not a constant, but it is running as

$$\alpha\left(q^2\right) \propto \left(\log\left(-q^2\right)\right)^{-1}.$$

as $q \to \infty$, the ultraviolet divergence.
- The divergences are removed by the physical parameter-fitting (m and e) from the *experiments*.
- The *ad hoc* Pauli-Villars cutoff Λ, for example, has been introduced for eliminating very large momentum contributions from the theory.
- In other words, the small distance scale physics are eliminated and replaced by those parameters.
- However, we do *not* have any precise information for short distance physics.

Renormalization group flows

- For a scaling parameter $b < 1$, but $b \approx 1$, we rescale distances and momenta in accoring to

$$k' = k/b, \quad x' = xb,$$

so that the variable k' is integrated over $|k'| < \Lambda$.

- The field is also rescaled as

$$\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi.$$

- Our model system with an effective Lagrangian

$$\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x' \left[\frac{1}{2}\left(\partial_\mu{}'\phi'\right)^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4\right]$$

will yield the rescaled parameters

$$\begin{aligned} m'^2 &= \left(m^2 + \Delta m\right)^2 (1+\Delta Z)^{-1} b^{-2} \longrightarrow m^{*2}, \\ \lambda' &= (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4} \longrightarrow \lambda^*. \end{aligned}$$

Renormalization scale

We introduce an arbitrary momentum scale M (*renormalization scale*) and impose the renormalization condition at a spacelike momentum p with $p^2 = -M^2$. Then we may have

$$\langle\Omega|\,\phi_0(p)\,\phi_0(-p)\,|\Omega\rangle = \frac{iZ}{p^2} \quad \text{at } p^2 = -M^2.$$

The n-point Green's function is defined by

$$G^{(n)}(x_1, \cdots, x_n) = \langle\Omega|\,\hat{T}\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle_{\text{connected}}\,.$$

If we shift M by δM, then correspondingly we obtain

$$\begin{aligned} M &\rightarrow M + \delta M, \\ \lambda &\rightarrow \lambda + \delta\lambda, \\ \phi &\rightarrow (1 + \delta\eta)\,\phi, \\ G^{(n)} &\rightarrow (1 + n\delta\eta)\,G^{(n)}. \end{aligned}$$

The Callan-Symanzik equation

If we think of $G^{(n)}$ as a function of M and λ, we can write as

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)}.$$

It is convenient to introduce dimensionless parameters

$$\beta \equiv \frac{M}{\delta M}\delta\lambda; \quad \gamma \equiv -\frac{M}{\delta M}\delta\eta,$$

so to arrive at the equation

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right] G^{(n)}\left(x_1, \cdots, x_n; M, \lambda\right) = 0.$$

Since G is renormalized, β and γ cannot depend on Λ, these functions cannot depend on M. We concluded that

$$\left[M\frac{\partial}{\partial M} + \beta\left(\lambda\right)\frac{\partial}{\partial\lambda} + n\gamma\left(\lambda\right)\right] G^{(n)}\left(\{x_i\}; M, \lambda\right) = 0.$$

This is known as the *Callan-Symanzik equation*.

Solutions of the Callan-Symanzik equations

The generic form of the two-point Green's function is

$G^{(2)}(p) =$ $+ \text{ loops } +$  $+ \cdots$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A \log \frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}\left(ip^2\delta_Z\right)\frac{i}{p^2} + \cdots$$

The M dependence comes entirely from the counterterm δ_Z. By neglecting the β term, we find

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z .$$

Because the counterterm must be

$$\delta_Z = A \log \frac{\Lambda^2}{M^2} + \text{finite},$$

to lowest order we have

$$\gamma = A.$$

Solutions of the Callan-Symanzik equations

In a similar manner we obtain

$$\beta(\lambda) = M \frac{\partial}{\partial M} \left(-\delta_\lambda + \frac{1}{2} \lambda \sum_i \delta_{Z_i} \right).$$

Since

$$\delta_\lambda = -B \log \frac{\Lambda^2}{M^2} + \text{finite},$$

to lowest order we have

$$\beta(\lambda) = -2B - \lambda \sum_i A_i.$$

So β and γ are not depending on the renormalization scale M.

The QED solutions

There is a γ term for each field and a β term for each coupling.

$$\left[M\frac{\partial}{\partial M}+\beta\left(e\right)\frac{\partial}{\partial e}+n\gamma_2\left(e\right)+m\gamma_3\left(e\right)\right]G^{(n,m)}\left(\{x_i\}\,;M,e\right)=0,$$

where n and m are, respectively, the number of electron and photon fieldsin $G^{(n,m)}$ and γ_2 and γ_3 are the rescaling functions of the electron and photon fields.

- $\beta \propto$ the shift in the coupling constant and
- $\gamma \propto$ the shift in the field renormalization,

when the renormalization scale M is increased.
Using the methods described before we obtain, to lowest order,

$$\beta(e)=\frac{e^3}{12\pi^2},\ \gamma_2(e)=\frac{e^2}{16\pi^2},\ \gamma_3(e)=\frac{e^2}{12\pi^2}.$$

Running coupling in QED

If $M \sim \mathcal{O}(m)$, then the renormalized value e_r is close to e. For the static potential $V(\mathbf{x})$, we have the Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta(e_r)\frac{\partial}{\partial e_r}\right] V(q; M, e_r) = 0.$$

Since the dimension of the Fourier transformed potential $V(q)$ is $(\mathrm{mass})^{-2}$, we trade M and q:

$$\left[q\frac{\partial}{\partial q} - \beta(e_r)\frac{\partial}{\partial e_r} + 2\right] V(q; M, e_r) = 0.$$

The potential will be in the form

$$V(q, e_r) = \frac{1}{q^2}\mathcal{V}(\bar{e}(q; e_r)),$$

where $\bar{e}(q)$ is the solution of the renormalization group equation

$$\frac{d}{d\log\left(\frac{q}{M}\right)}\bar{e}(q; e_r) = \beta(\bar{e}), \quad \bar{e}(M; e_r) = e_r.$$

Running coupling in QED

Since the potential, in leading order, is

$$V(q) \approx \frac{e^2}{q^2},$$

we can identify $\mathcal{V}(\bar{e}) = \bar{e}^2 + \mathcal{O}(\bar{e}^4)$. We immediately obtain

$$V(q, e_r) = \frac{\bar{e}^2(q; e_r)}{q^2}.$$

By solving the renormalization group equation for $\bar{e}$ and using $\beta(e) = e^3/12\pi^2$, we find

$$\frac{12\pi^2}{2}\left(\frac{1}{{e_r}^2} - \frac{1}{\bar{e}^2}\right) = \log\frac{q}{M}.$$

This simplifies to

$$\bar{e}^2(q) = \frac{{e_r}^2}{1 - ({e_r}^2/6\pi^2)\log(q/M)}.$$

Running coupling in QED

By setting $M^2 = \exp(5/3)\, m^2$ and $e_r \approx e$, with $\alpha = \frac{e^2}{4\pi}$, we reproduce

$$\bar{\alpha}(q) = \frac{\alpha}{1 - \left(\frac{\alpha}{3\pi}\right) \log\left(-\frac{q^2}{Am^2}\right)}, \quad A = \exp(5/3).$$

There is a renormalization scale M, which replaces the *ad hoc* Pauli-Villars cutoff Λ.

$\star$ The electric charge is the result of the virtual vacuum polarization by the existence of interacting electron.

Evolution of mass

If $\mathcal{L}_M$ is the massless Lagrangian renormalized at the scale M, the new massive Lagrangian will be in the form

$$\mathcal{L} = \mathcal{L}_M - \frac{1}{2} m^2 {\phi_M}^2 .$$

We treat mass term by replacing $m^2 \to \rho_m M^2$ and expanding the Lagrangian about the free field one $\mathcal{L}_0$ reads:

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2} \rho_m M^2 {\phi_M}^2 - \frac{1}{4} \lambda M^{4-d} {\phi_M}^4 ,$$

which is the Landau-Ginzburg theory for ***ferromagnetism!***
The Callan-Symanzik equation will give us

$$\beta = -\left(4 - d\right) \lambda + \frac{3\lambda^2}{16\pi^2}$$

and for the condition $\beta = 0$

$$\lambda_* = \frac{16\pi^2}{3} \left(4 - d\right) .$$

Mass from a phase transition

The corresponding renormalization group equation would be

$$\frac{d}{d\log p}\bar{\rho}_m = \left[-2 + \gamma_{\phi^2}\left(\bar{\lambda}\right)\right]\bar{\rho}_m.$$

The solution is, for the coupling $\bar{\lambda} = \lambda_*$,

$$\bar{\rho}_m = \rho_m \left(\frac{M}{p}\right)^{2-\gamma_{\phi^2}(\lambda_*)} .$$

The solution gives a nontrivial relation

$$\xi \sim {\rho_m}^{-\nu},$$

where the exponent ν is given formally by the expression

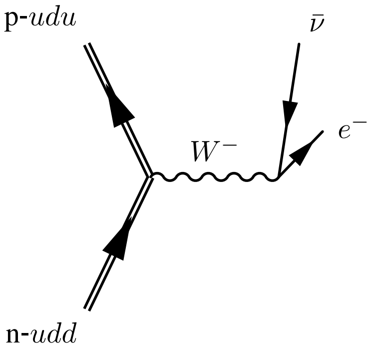
$$\nu = \frac{1}{2 - \gamma_{\phi^2}\left(\lambda_*\right)},$$

explicitly, the Wilson-Fisher relation in statistical physics

$$\nu^{-1} = 2 - \frac{1}{3}\left(4 - d\right).$$

β-decay

- The radioactivity discovered by Becquerel is β-decay.
- This is the neutron decay process: $\mathrm{n} \to \mathrm{p} + \bar{\nu} + e^-$,



- β-decay violates the CP gauge symmetry.
- A non-Abelian gauge theory, $SU(2) \times U(1)$, is required.
- ***No*** massive bosons and fermions are allowed to satisfy the $SU(2) \times U(1)$ gauge symmetry.

Massless Dirac field

Let a Dirac field ψ is *massless*, but it is a doublet of Dirac fields

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

The kinetic energy term may be written as

$$\mathcal{L} = {\psi_L}^\dagger i\bar{\sigma} \cdot \partial \psi_L + {\psi_R}^\dagger i\sigma \cdot \partial \psi_R.$$

The left-handed fields may coupled to a *non-Abelian gauge field* ${A^a}_\mu$, which defines the corresponding field tensor as

$${F^a}_{\mu\nu} = \partial_\mu {A^a}_\nu - \partial_\nu {A^a}_\mu + g f^{abc} {A^b}_\mu {A^c}_\nu,$$

through the minimal substitution $D_\mu = \partial_\mu - ig{A^a}_\mu {t^a}_r$ to yield

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \left(\partial_\mu - ig{A^a}_\mu {t^a}_r \left(\frac{1-\gamma^5}{2}\right)\right)\psi.$$

Here t^a follows the commutation relation $[t^a, t^b] = if^{abc}t^c$.

Higgs coupling

- We may assign the left-handed components of quarks and leptons to doublets of an $SU(2)$ gauge symmetry like

$$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}, \quad L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}.$$

- Since these fields are massless, we introduce a $U(1)$ gauge symmetric field ϕ, which is known as ***Higgs field***,

$$D_\mu \phi = \left(\partial_\mu - igA^a{}_\mu \tau^a\right)\phi,$$

where $\tau^a = \frac{\sigma^a}{2}$.
- If the vacuum expectation value of ϕ has *broken* symmetry

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix},$$

then the gauge boson masses arise from

$$|D_\mu \phi|^2 = \frac{1}{2} g^2 \left(0 \;\; v\right) \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} A^a{}_\mu A^{b\mu} + \cdots$$

Higgs mechanism

After a symmetrization we find the mass term

$$\Delta\mathcal{L} = \frac{g^2 v^2}{8} A^a{}_\mu A^{a\mu}.$$

All three gauge bosons receive the mass $m_A = \frac{gv}{2}$.
When Higgs field transforms under ϕ follows $SU(2) \times U(1)$ gauge symmtry,

$$\phi \to e^{i\alpha^a \tau^a} e^{i\beta/2} \phi,$$

two bosons acquire masses and one boson remains massless:

$W^\pm$ $m_{W^\pm} = \frac{gv}{2}$,

Z^0 $m_{Z^0} = \sqrt{g^2 + g'^2}\frac{v}{2}$,

A $m_A = 0$.

Higgs mechanism

Similarly, the electron fields $\bar{e}_L$ and e_R follows the mass term

$$\Delta\mathcal{L}_e = -\frac{1}{\sqrt{2}}\lambda_e v \bar{e}_L e_R + \text{ h.c.} + \cdots ,$$

by which the massless electron acquires mass $m_e = \frac{1}{\sqrt{2}}\lambda_e v$.

$\star$ The electron mass is the result of the spontaneous continuous symmetry breaking of Higgs field.

Summary

- Relativistic quantum field theory
  - Intrinsic spin
  - Pauli's principle
- Field quantization
  - Klein-Gordon fields
  - Dirac fields
  - Propagator and causality
- Interacting field theory
  - S-matrix theory
  - Perturbation expansion
  - Photon as gauge particle
  - Elementary processes
- Renormalization $\Rightarrow$ Physically observed parameters:
  - spin-magnetic momentum (definite),
  - electron mass (cancelled divergences),
  - electric charge of electron (leaving divergence).
- Renormalization Group and Higgs mechanism.
  - The origin of the charge of electron,
  - The origin of the electron mass.

James Clerk Maxwell

The work of James Clerk Maxwell changed the world forever.
by Albert Einstein